Online Appendix to

Born Free

by
Nils-Petter Lagerlöf
lagerlof@econ.yorku.ca

Equations in Arabic numerals below refer to those in the original paper. Equations in (small) Roman numerals refer those appearing in these very notes.

1 The Euler equation

The elite's problem to maximize (11) subject to $c_t^E = (1 - s_t)\,\pi_t$, (8), and (10) can be written recursively as

$$V(\pi_t) = \max_{s_t \in [0,1]} \left\{ \ln\left([1 - s_t]\pi_t\right) + \beta V(\widehat{\pi}_{t+1} + s_t\gamma\pi_t w_{t+1}^S) \right\}, \tag{i}$$

where we have substituted $\widehat{\pi}_{t+1} + w_{t+1}^S s_t\gamma\pi_t$ for π_{t+1}, using (8) and (10). The first-order condition states that optimal s_t must satisfy

$$\frac{1}{1 - s_t} = \beta V'(\pi_{t+1})\gamma\pi_t w_{t+1}^S, \tag{ii}$$

where $\pi_{t+1} = \widehat{\pi}_{t+1} + w_{t+1}^S s_t\gamma\pi_t$.

Evaluating the expression that is maximized in (i) at optimal s_t, and applying the Envelope Theorem, gives $V'(\pi_t)$ as

$$V'(\pi_t) = \frac{1}{\pi_t} + \beta V'(\pi_{t+1}) s_t\gamma w_{t+1}^S, \tag{iii}$$

where, recall again, $\pi_{t+1} = \widehat{\pi}_{t+1} + w_{t+1}^S s_t\gamma\pi_t$. Multiplying and dividing the right-hand side of (iii) by π_t gives

$$\begin{aligned} V'(\pi_t) &= \tfrac{1}{\pi_t} + \beta V'(\pi_{t+1})\gamma\pi_t w_{t+1}^S \left(\tfrac{s_t}{\pi_t}\right) \\ &= \tfrac{1}{\pi_t} + \tfrac{1}{1-s_t}\tfrac{s_t}{\pi_t} \\ &= \tfrac{1}{\pi_t}\left(\tfrac{1}{1-s_t}\right) = \tfrac{1}{c_t^E}, \end{aligned} \tag{iv}$$

where the second equality uses $\beta V'(\pi_{t+1})\gamma\pi_t w_{t+1}^S = 1/(1 - s_t)$, which follows from the first-order condition in (ii), and the last equality follows from $c_t^E = (1 - s_t)\,\pi_t$.

To arrive at (12), first divide (ii) by π_t, then recall $c_t^E = (1 - s_t)\,\pi_t$ again, and finally forward (iv) one period to substitute $V'(\pi_{t+1})$ for $1/c_{t+1}^E$.

2 Migration and the rural-urban worker ratio

For urban workers consumption is $c_t^F = w_t^F - (q/\varepsilon)n_t^F = (1-\widetilde{\gamma})w_t^F$, and fertility $n_t^F = \varepsilon\widetilde{\gamma}w_t^F/q$. Using (1) this gives

$$u_t^F = \ln\left[(1-\widetilde{\gamma})^{(1-\widetilde{\gamma})}(\widetilde{\gamma})^{\widetilde{\gamma}}\right] + \ln(w_t^F) - \widetilde{\gamma}\ln(q) + \widetilde{\gamma}\ln(\varepsilon). \tag{v}$$

For rural workers consumption is $c_t^L = w_t^L - qn_t^L = (1-\widetilde{\gamma})w_t^L$, and fertility $n_t^L = \widetilde{\gamma}w_t^L/q$. Using (1) this gives

$$u_t^L = \ln\left[(1-\widetilde{\gamma})^{(1-\widetilde{\gamma})}(\widetilde{\gamma})^{\widetilde{\gamma}}\right] + \ln(w_t^L) - \widetilde{\gamma}\ln(q). \tag{vi}$$

Setting $u_t^F = u_t^L$ now gives (15).

Next we derive (16). Using (15), and letting $R_t = L_t/F_t$, we can rewrite (14) as

$$F_{t+1} = \gamma w_t^L F_t\left[\varepsilon^{1-\gamma q} + (1-\theta_{t+1})R_t\right]. \tag{vii}$$

Then $R_t = L_t/F_t$ and (13) give

$$L_{t+1} = \theta_{t+1}\gamma w_t^L R_t F_t. \tag{viii}$$

Now dividing (viii) by (vii) gives (16), with $R_t = L_t/F_t$.

3 Slave per-capita incomes in the transition

Let the free-worker non-migration rate be constant at $\overline{\theta}^H$ for $t \in [0, \tau-1]$, and then fall to $\overline{\theta}^L < \overline{\theta}^H$ for $t \geq \tau$. Also, assume that the change in $\overline{\theta}$ is unanticipated (meaning that the probability of $\overline{\theta}$ changing is close to zero), and that the economy is in steady-state up until period $\tau - 1$. Then S_τ (the slave population in period τ) equals the steady state level associated with $\overline{\theta} = \overline{\theta}^H$. Moreover, for all $t \in [0, \tau-1]$, it holds that slave fertility is at replacement, $n_t^S = 1$, and slave per-capita income is at its steady-state level, $y_t^S = 1/\gamma$; recall (2).

We also know from Result 1 that the new steady state, associated with $\overline{\theta} = \overline{\theta}^L$, has larger slave population. Thus, for some sufficiently large $T > \tau$, it must hold that $S_T > S_\tau$. Since $S_{t+1} = S_t n_t$ for all $t \geq 0$, we can write S_T as

$$S_T = S_\tau \prod_{t=\tau}^{T-1} n_t^S. \tag{ix}$$

From $S_T > S_\tau$, we now see that the (geometric) average of n_t^S, taken over some arbitrarily long period starting in period τ, must exceed one. That is, by logging (ix) we see that for sufficiently large $T > \tau$ it holds that

$$\ln(S_T) - \ln(S_\tau) = \frac{\sum_{t=\tau}^{T-1}\ln(n_t^S)}{T-\tau} > 0. \tag{x}$$

Since $n_t^S = \gamma y_t^S$, this implies that for $t \in [\tau, T]$ slave per-capita incomes, y_t^S, are on average above the level they were at for $t \in [0, \tau - 1]$, namely $1/\gamma$.

4 Determining $\overline{\theta}$ with endogenous migration

4.1 The shapes of $\mathcal{L}^I(\theta)$ and $\mathcal{L}^{II}(\theta, A^U)$

Letting

$$\Omega(\theta) = \frac{\eta^{\frac{1}{1-\rho}}\theta^{\frac{\rho}{1-\rho}}}{\beta^{\frac{\rho}{1-\rho}} + \eta^{\frac{1}{1-\rho}}\theta^{\frac{\rho}{1-\rho}}} \in (0,1), \tag{xi}$$

and differentiating (23), it can be seen that

$$\frac{\partial \ln\left[\mathcal{L}^I(\theta)\right]}{\partial\theta} = \frac{1}{(1-\rho)\theta}\left[1 - \left(\frac{\rho-\alpha}{1-\alpha}\right)\Omega(\theta)\right] > 0, \tag{xii}$$

where we recall that $0 < \alpha < \rho \leq 1$. Then (25) shows that, for $\theta > \varepsilon^{1-\gamma q}$, it holds that

$$\frac{\partial \ln\left[\mathcal{L}^{II}(\theta, A^U)\right]}{\partial\theta} = \frac{1}{(1-\delta)\theta} + \frac{1-\varepsilon^{1-\gamma q}}{\left(\theta - \varepsilon^{1-\gamma q}\right)(1-\theta)} > 0. \tag{xiii}$$

4.2 Existence of $\overline{\theta}$

Let

$$D(\theta, A^U) = \ln\left[\mathcal{L}^{II}(\theta, A^U)\right] - \ln\left[\mathcal{L}^I(\theta)\right]. \tag{xiv}$$

Now $\overline{\theta}$ is defined from $D(\overline{\theta}, A^U) = 0$. Using (23) and (25), it is easy to verify that (a) $\lim_{\theta\to\varepsilon^{1-\gamma q}} D(\theta, A^U) = -\infty$, since $\lim_{\theta\to\varepsilon^{1-\gamma q}} \mathcal{L}^{II}(\theta, A^U) = 0$, and $\lim_{\theta\to\varepsilon^{1-\gamma q}} \mathcal{L}^I(\theta) > 0$; and (b) $\lim_{\theta\to 1} D(\theta, A^U) = \infty$, since $\lim_{\theta\to 1} \mathcal{L}^{II}(\theta, A^U) = \infty$, and $\lim_{\theta\to 1} \mathcal{L}^I(\theta)$ is finite. From the continuity of $D(\theta, A^U)$, it follows that that some $\overline{\theta} \in (\varepsilon^{1-\gamma q}, 1)$ exists, such that $D(\theta, A^U) = 0$. This proves existence.

4.3 Uniqueness of $\overline{\theta}$

To prove uniqueness we must show that $D(\theta, A^U)$ is strictly increasing in θ. A sufficient conditions for this to hold will be seen to be that $\delta \geq (2\rho - 1)/\rho$. Using (xii) and (xiii), some algebra shows that

$$\begin{aligned}
\frac{\partial D(\theta, A^U)}{\partial\theta} &= \frac{\partial \ln\left[\mathcal{L}^{II}(\theta, A^U)\right]}{\partial\theta} - \frac{\partial \ln\left[\mathcal{L}^I(\theta)\right]}{\partial\theta} \\
&= \frac{1}{(1-\rho)\theta}\left[\left(\frac{\delta-\rho}{1-\delta}\right) + \left(\frac{\rho-\alpha}{1-\alpha}\right)\Omega(\theta) + \left(\frac{\theta}{\theta-\varepsilon^{1-\gamma q}}\right)\left(\frac{1-\varepsilon^{1-\gamma q}}{1-\theta}\right)(1-\rho)\right] \\
&> \frac{1}{(1-\rho)\theta}\left[\left(\frac{\delta-\rho}{1-\delta}\right) + (1-\rho)\right],
\end{aligned} \tag{xv}$$

where the last inequality uses $\rho > \alpha$, $\Omega(\theta) > 0$, $\theta/(\theta - \varepsilon^{1-\gamma q}) > 1$, and $(1-\varepsilon^{1-\gamma q})/(1-\theta) > 1$. We thus see from (xv) that a sufficient condition for $D_\theta(\theta, A^U) > 0$ is that $(\delta - \rho) + (1 - \delta)(1 - \rho) \geq 0$, which can be written $\delta \geq (2\rho - 1)/\rho$.

4.4 Showing that $\overline{\theta}$ is a decreasing function of A^U

Recall from (xv) that $\partial D(\theta, A^U)/\partial\theta > 0$, and note from (25) that $\partial D(\theta, A^U)/\partial A^U > 0$. We can now use implicit differentiation of (xiv) to see that

$$\frac{d\overline{\theta}}{dA^U} = -\frac{\frac{\partial D(\theta, A^U)}{\partial A^U}}{\frac{\partial D(\theta, A^U)}{\partial \theta}} < 0. \tag{xvi}$$

5 Barriers to mobility of free workers

Here we consider a setting were the elite can erect barriers to free worker's mobility. For rural workers migrating to the urban sector, utility is now given by (v), minus a utility loss of $\ln(\chi_t)$:

$$u_t^F = \ln\left[(1-\widetilde{\gamma})^{(1-\widetilde{\gamma})}(\widetilde{\gamma})^{\widetilde{\gamma}}\right] + \ln(w_t^F) - \widetilde{\gamma}\ln(q) + \widetilde{\gamma}\ln(\varepsilon) - \ln(\chi_t). \tag{xvii}$$

Setting u_t^L in (vi) equal to u_t^F in (xvii) gives

$$w_t^L = \frac{\varepsilon^{\gamma q} w_t^F}{\chi_t}, \tag{xviii}$$

where $\gamma q = \widetilde{\gamma}$. A higher χ_t thus implies lower wages for free workers.

Let the cost of barriers to the elite be $k\chi_t$, for some $k > 0$. Taking w_t^S and w_t^F as given, the elite (collectively) set χ_t in each period to maximize the land income of the representative elite agent, given as $\widehat{\pi}_t$ in (9). This can be rewritten as $\widehat{\pi}_t = (1-\alpha)A^R Z_t^{\frac{\alpha}{\rho}}$, where $Z_t = S_t^\rho + \eta L_t^\rho$. Using (A6) this gives

$$\widehat{\pi}_t = (1-\alpha)\alpha^{\frac{\rho}{1-\alpha}}\left(A^R\right)^{\frac{1-\alpha+\rho}{1-\alpha}}\left[\left(\frac{1}{w_t^S}\right)^{\frac{\rho}{1-\rho}} + \eta^{\frac{1}{1-\rho}}\left(\frac{\chi_t}{\varepsilon^{\gamma q} w_t^F}\right)^{\frac{\rho}{1-\rho}}\right]^{\left(\frac{1-\rho}{\rho}\right)\left(\frac{\alpha}{1-\alpha}\right)}, \tag{xix}$$

where we have also used (xviii). The elite's optimal choice of χ_t is now given by

$$\frac{\partial\widehat{\pi}_t}{\partial\chi_t} = k, \tag{xx}$$

where the second-order condition can be seen to hold because the expression in square brackets in (xix) is concave in χ_t, i.e., $\partial^2\widehat{\pi}_t/\partial\chi_t^2 < 0$. [To see this, define $\widehat{\rho} = \rho/(1-\rho)$ and $\widehat{\alpha} = \alpha/(1-\alpha)$, and note that $\alpha < \rho$ implies $\widehat{\alpha} < \widehat{\rho}$; cf. Section B in the appendix of the paper.]

It can be seen from (xix) that a fall in w_t^S (meaning that slaves become less expensive) leads to a fall in $\partial\widehat{\pi}_t/\partial\chi_t$. From (xx) and $\partial^2\widehat{\pi}_t/\partial\chi_t^2 < 0$ follows that this leads to a fall in the optimal choice of χ_t. Note that a slave-free society amounts to letting $w_t^S \to \infty$.

It is also easy to see that a rise in w_t^F leads to a fall in the optimal choice of χ_t, as long as slave labor is available (w_t^S is finite). That is, a rise in the outside wage for free workers induces the elite to substitute to slave labor, thus investing less in restricting free labor mobility.