# Supplementary Notes to Pacifying Monogamy

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## 1 Introduction

These notes consider an extended version of the model of pacifying monogamy set up by Lagerlöf (2009), referred to below as "the paper."

In the framework presented in the paper, we treated the distribution of power between the King and his subjects as exogenous. Thus, the only means by which a King could pacify other males was by sharing women (i.e, constraining polygyny). Here we consider a setting where the King is also allowed to allocate, or delegate, power to the subjects as a means to pacify them, but at the cost of raising the risks of a successful rebellion.

## 2 An extended model

We now let $q$ denote the power held by the subjects, so that $1 - q$ is the King's power; $q$ thus replaces $\omega$ in the paper. We can think of $1 - q$ as the rate at which the King chooses to tax his subjects, so that less taxation is equivalent to giving the subjects more power.

We assume that the King must give the subjects a minimum amount of power, which we now denote $\omega$. One interpretation is that $1-\omega$ is the highest rate at which the King can tax his subjects, e.g. because the subjects can hide a fraction $\omega$ of total resources. The King thus chooses $q$ on $[\omega, 1]$, and if he sets $q = \omega$ all results become identical to when we assumed that power could not be shared. The exercise that we undertake in these notes is to find sufficient conditions for the King to choose $q = \omega$.

The idea is that more empowered subjects have a greater probability of success in rebellion. We now let the rebellion success function in (5) in the paper take this form:

$$G(R, q) = \min\{1, \eta(q)R\}, \tag{1}$$

where $\eta'(q) > 0$. To keep close to the original notation we let

$$\eta(\omega) = \varepsilon, \tag{2}$$

and we maintain the assumption that $\varepsilon P > 1$. From $\eta'(q) > 0$ and $q \in [\omega, 1]$ it follows that $\eta(q)P > 1$.

As in the paper, we can make different assumptions about the King's ability to write binding rules. Either the King's actions commit also a potentially successful rebel, or they do not.

## 2.1 Optimal constraints on power sharing

Consider first an approach similar to that in Section 3.3 in the paper, under which the King chooses a rule which constrains also a potentially successful rebel's choices. This rule now pins down both $z$ and $q$.

We begin by redefining the function $\phi$ in (11) in the paper as follows:

$$\phi(z, q) = [1 - \eta(q)P]\,(1 - q)^{\rho}(1 - z)^{1-\rho} + q^{\rho}z^{1-\rho}. \tag{3}$$

Note that if we set $q = \omega$, recalling (2) above, then (3) boils down to (11) in the paper. Next define

$$\gamma(q) = \frac{[\eta(q)P]^{\frac{1}{1-\rho}}\left(\frac{1-q}{q}\right)^{\frac{\rho}{1-\rho}}}{1 + [\eta(q)P]^{\frac{1}{1-\rho}}\left(\frac{1-q}{q}\right)^{\frac{\rho}{1-\rho}}}, \tag{4}$$

and

$$\delta(q)=\frac{[\eta(q)P-1]^{\frac{1}{1-\rho}}\left(\frac{1-q}{q}\right)^{\frac{\rho}{1-\rho}}}{1+[\eta(q)P-1]^{\frac{1}{1-\rho}}\left(\frac{1-q}{q}\right)^{\frac{\rho}{1-\rho}}}, \tag{5}$$

where we note from (7) and (8) in the paper, and (2) above, that $\gamma(\omega)=\overline{z}$ and $\delta(\omega)=\widehat{z}$.

Clearly, Propositions 2, 3, and 4 in the paper still apply, but with $q$ replacing $\omega$ and $\eta(q)$ replacing $\varepsilon$. Analogously to (12) in the paper the King's expected number of offspring, now denoted $k(z,q)$, becomes:

$$k(z,q)=\begin{cases}(1-z)^{1-\rho}(1-q)^{\rho}A^{\rho}P^{1-\rho} & \text{if } z\in[\gamma(q),1],\\ \\ \phi(z,q)A^{\rho}P^{1-\rho} & \text{if } z\in[\delta(q),\gamma(q)],\\ \\ 0 & \text{if } z\in[0,\delta(q)].\end{cases} \tag{6}$$

The King maximizes $k(z,q)$ over $z$ and $q$ in the choice set

$$\begin{aligned}\Omega &= \{(z,q)\in[0,1]^2 : z\geq q, q\geq\omega\}\\ &= \{(z,q)\in[\omega,1]^2 : z\geq q\},\end{aligned} \tag{7}$$

where $z\geq q$ follows from the women's participation constraint (recall Proposition 1 in the paper). We can now define the King's optimal choice of $z$ and $q$, denoted $z^{OPT}$ and $q^{OPT}$, as

$$\left(z^{OPT},q^{OPT}\right)=\arg\max_{(z,q)\in\Omega}k(z,q). \tag{8}$$

Analogous to Proposition 5, it is easy to see from (6) that (if the solution is interior) $k(z,q)$ is maximized when $z=\gamma(q)$, i.e., when the subjects are fully pacified. The King's expected fertility thus equals $k(\gamma(q),q)$. Substituting $z=\gamma(q)$ into $k(z,q)$ the King's expected fertility becomes $A^{\rho}P^{1-\rho}$ times a function of $q$ only, call it $\pi(q)$. A bit of algebra shows that

$$\pi(q)\equiv\phi(\gamma(q),q)=\left[\frac{(1-q)^{\frac{\rho}{1-\rho}}}{1+[\eta(q)P]^{\frac{1}{1-\rho}}\left(\frac{1-q}{q}\right)^{\frac{\rho}{1-\rho}}}\right]^{1-\rho}. \tag{9}$$

The task is thus to find some $q$ on $[\omega, 1]$ that maximizes $\pi(q)$. If $\pi(q)$ is decreasing in $q$ on $[\omega, 1]$, then the King's expected number of children is maximized at $q = \omega$. A sufficient condition for this to be the case is that the elasticity of $\eta(q)$ is big enough. Let this elasticity be denoted $r(q)$, i.e.,

$$r(q) = \frac{\eta'(q)q}{\eta(q)}. \tag{10}$$

We can now impose the following assumption:

**Assumption 1** *$r(q)(1-q) > \rho$ for all $q \in [\omega, 1)$.*

If $\eta(q)$ is such that Assumption 1 holds, then the King's expected fertility is maximized when the power allocated to subjects, $q$, equals its minimum level, $\omega$. Let us sum this up in a proposition.

**Proposition 1** *Under Assumption 1 in the paper ($\omega > 1/2$) and Assumption 1 above, the result in Proposition 5 in the paper holds: $q^{OPT} = \omega$, and $z^{OPT}$ is given by (14) in the paper.*

Intuitively, the function $\eta(q)$ measures how much the King's vulnerability increases when he allocates more power to the subjects. If $\eta(q)$ is very elastic, sharing power is costly, in the sense that it lowers the King's survival probabilities a lot. With sufficient elasticity (as stated in Assumption 1) it is optimal for the King not to share any power, and instead pacify the subjects only by sharing women. All results are identical to the setting in the paper where we simply assumed that the King could not share power.

One example of a functional form for $\eta(q)$ that satisfies Assumption 1 is

$$\eta(q) = \left(\frac{q}{1-q}\right)^{\kappa}, \tag{11}$$

for some $\kappa > \rho$. Note that, since the probability of ousting the King depends on the ratio of the powers held by the contestant parties, the functional form in (11) is similar to a standard Tullock-like contest function.

## 2.2 Equilibrium constraints on power sharing

Consider next the approach used in Section 3.4 in the paper, under which the King cannot constrain the potentially successful rebel. Now we need to restate the payoffs to the subjects of rebelling and staying peaceful. Analogous to (15) in the paper we can write:

$$m(S; z, \widetilde{z}, q, \widetilde{q}, x) = \begin{cases} \frac{z^{1-\rho} q^{\rho}}{x} \left(\frac{A}{P}\right)^{\rho} & \text{if } S = \mathcal{P}, \\ \left[\frac{G([1-x]P, q)}{(1-x)P}\right] (1-\widetilde{z})^{1-\rho} (1-\widetilde{q})^{\rho} A^{\rho} P^{1-\rho} & \text{if } S = \mathcal{R}. \end{cases} \tag{12}$$

where $G(R, q)$ is now given by (1). The power that the subjects are allocated by the incumbent King is denoted $q$, whereas the power a successful rebel would allocate to his subjects is here denoted $\widetilde{q}$.

Going through the same steps as in Propositions 6 to 8 in the paper we can derive an expression for the incumbent King's expected number of children, now as a function of four variables: women and power shared by the incumbent King ($z$ and $q$), and women and power shared by a successful rebel ($\widetilde{z}$ and $\widetilde{q}$). Begin by redefining the function $\lambda$ in (20) in the paper as follows:

$$\lambda(z, \widetilde{z}, q, \widetilde{q}) = \left[1 - \eta(q)P + \left(\frac{q}{1-\widetilde{q}}\right)^{\rho} \left(\frac{z}{1-\widetilde{z}}\right)^{1-\rho}\right] (1-z)^{1-\rho}(1-q)^{\rho}, \tag{13}$$

where we note that setting $q = \widetilde{q} = \omega$ brings back (20) in the paper.

Next redefine the functions $\beta$ and $\alpha$ in (16) and (17) in the paper as follows:

$$\beta(\widetilde{z}, q, \widetilde{q}) = (1-\widetilde{z}) \left[\eta(q)P\right]^{\frac{1}{1-\rho}} \left(\frac{1-\widetilde{q}}{q}\right)^{\frac{\rho}{1-\rho}}, \tag{14}$$

and

$$\alpha(\widetilde{z}, q, \widetilde{q}) = (1-\widetilde{z}) \left[\eta(q)P - 1\right]^{\frac{1}{1-\rho}} \left(\frac{1-\widetilde{q}}{q}\right)^{\frac{\rho}{1-\rho}}. \tag{15}$$

Analogously to Proposition 8 in the paper we can now write the incumbent

King's expected number of children as:

$$K(z,\widetilde{z},q,\widetilde{q}) = \begin{cases} (1-z)^{1-\rho}(1-q)^{\rho}A^{\rho}P^{1-\rho} & \text{if } z \in [\beta(\widetilde{z},q,\widetilde{q}),1], \\ \\ \lambda(z,\widetilde{z},q,\widetilde{q})A^{\rho}P^{1-\rho} & \text{if } z \in [\alpha(\widetilde{z},q,\widetilde{q}),\beta(\widetilde{z},q,\widetilde{q})], \\ \\ 0 & \text{if } z \in [0,\alpha(\widetilde{z},q,\widetilde{q})]. \end{cases} \tag{16}$$

The King now chooses $z$ and $q$ to maximize $K(z,\widetilde{z},q,\widetilde{q})$, subject to $(z,q) \in \Omega$ in (7), taking $\widetilde{z}$ and $\widetilde{q}$ as given. His optimal choices can then be defined as best-response functions of $\widetilde{z}$ and $\widetilde{q}$. Applying the same notation as in (22) in the paper we can denote the incumbent King's optimal choice by the function $\Psi$, i.e.,

$$\Psi(\widetilde{z},\widetilde{q}) = \arg\max_{(z,q)\in\Omega} K(z,\widetilde{z},q,\widetilde{q}). \tag{17}$$

We define the equilibrium $z$ and $q$, denoted $z^{EQ}$ and $q^{EQ}$, as a fixed point to $\Psi(\widetilde{z},\widetilde{q})$, that is:

$$(z^{EQ},q^{EQ}) = \Psi(z^{EQ},q^{EQ}). \tag{18}$$

We can now impose a condition on $\eta(q)$, slightly weaker than that in Assumption 1, that is sufficient to ensure that the King chooses not to share any power.

**Assumption 2** *$r(q)(1-q)/q > \rho$ for all $q \in [\omega,1)$.*

If $\eta(q)$ is such that Assumption 2 holds, then the equilibrium level of $q$ equals its minimum level, $\omega$. Let us sum this up in a proposition.

**Proposition 2** *Under Assumptions 1 and 2 in the paper, and 2 above, the result in Proposition 9 in the paper holds. That is, $q^{EQ} = \omega$ and $z^{EQ}$ is given as follows:*
*(a) If $\varepsilon P \leq \omega/(1-\omega)$, then $z^{EQ} = \omega$.*
*(b) If $\varepsilon P \geq \omega/(1-\omega)$, then $z^{EQ}$ is defined from $J(z^{EQ},\omega,P) \equiv 0$, where*

$$J(z,q,P) = (2z-1)\left(\frac{q}{1-q}\right)^{\rho} - (\varepsilon P - 1)\, z^{\rho}(1-z)^{1-\rho}. \tag{19}$$

The conditions in Assumptions 1 and 2 are not comparable, since they are only sufficient and not necessary. However, it can be shown that the result of non-delegation of power holds under weaker conditions when the incumbent King cannot constrain his subjects compared to when he can. This makes intuitive sense, since the King's incentive to delegate power partly comes from constraining potentially successful rebels, which makes it less valuable for them to become the new King, thus reducing the risk of rebellion.

## 2.3 Discussion

The result that the King finds it optimal not to share power hinges on the assumption that doing so carries a cost in terms of increased risks of a successful rebellion. In other words, power is treated as an input in the rebellion success function. One could consider a model where women themselves constitute such an input. The King would then be less inclined to share women because doing so would increase the risk of a successful rebellion. In our model this issue does not arise due to how the events are timed: the allocation of women takes place after the competition for power. More generally, the empirical observation that men are more violent than women would also make such a mechanism less plausible.

For comparison, it is interesting to note how the results are altered if sharing power carries no cost in terms of increased risk of a successful rebellion. Analytically, this amounts to letting $\eta(q)$ be constant, i.e., setting $\eta'(q) = 0$. Under the optimal approach, corresponding to that in Section 3.3, the King chooses $q$ on $[\omega, 1]$ to maximize $\pi(q)$ in (9). After some algebra, the optimal shares of power and women allocated to the subjects can be seen to equal

$$z^{OPT} = q^{OPT} = \begin{cases} \omega & \text{if } \varepsilon P \leq \frac{\omega}{1-\omega}, \\ \frac{\varepsilon P}{1+\varepsilon P} & \text{if } \varepsilon P \geq \frac{\omega}{1-\omega}, \end{cases} \tag{20}$$

where we have set $\eta(q) = \varepsilon$, and recall that $z^{OPT} = \gamma(q^{OPT})$ for $\varepsilon P \geq \omega/(1-\omega)$. In a sense, the result that the King can find it optimal to share women ($z^{OPT} > \omega$) still holds; the news here is that he shares power as well, and in equal proportions. He also shares more women and power when the population ($P$) is large, so the scale effect by which population growth generates a transition to more constrained polygyny still holds.

However, since $z^{OPT} = q^{OPT}$ the King marries as many women as he can attract, given the amount of power he keeps for himself. In that sense, the constrained polygyny result goes away.

# APPENDIX

## A Proofs

*Proof of Proposition 1*: Start by finding the optimal choice of $z$, holding $q$ fixed at some $q^{OPT} \in [\omega, 1]$. Consider first the case when $\gamma(q^{OPT}) \geq \omega$. Since $\gamma(q^{OPT})$ takes the place of $\overline{z}$ we understand from the proof of Proposition 5 in the paper that $z^{OPT} = \gamma(q^{OPT})$. To find $q^{OPT}$ we substitute $z = \gamma(q)$ into $k(z, q) = \phi(z, q)A^{\rho}P^{1-\rho}$, to see that the maximization problem boils down to maximizing $\pi(q) \equiv \phi(\gamma(q), q)$ over $q \in [\omega, 1]$. Using (3), (4), and some algebra, it can be seen that $\pi(q)$ equals the expression in (9).

If $\pi(q)$ is decreasing in $q$ on the whole interval $[\omega, 1]$, then $q^{OPT} = \omega$. Since the numerator is decreasing in $q$, a sufficient condition for $\pi(q)$ to be decreasing in $q$ is that the denominator is increasing in $q$, which amounts to

$$\frac{\partial}{\partial q}\left[\eta(q)\left(\frac{1-q}{q}\right)^{\rho}\right] = \left(\frac{1-q}{q}\right)^{\rho}\frac{\eta(q)}{q}\left[r(q) - \frac{\rho}{1-q}\right] > 0, \qquad \text{(A1)}$$

were $r(q) = \eta'(q)q/\eta(q)$. The inequality in (A1) holds under Assumption 1, implying that $q^{OPT} = \omega$. Using $z^{OPT} = \gamma(q^{OPT})$, and (4) above, we see that $z^{OPT} = \gamma(\omega) = \overline{z}$, where we recall the definition of $\overline{z}$ in (7) in the paper.

We must also verify that $z^{OPT} = \gamma(\omega) = \overline{z} \geq \omega$ holds. Some algebra verifies that $\gamma(\omega) \geq \omega$ is equivalent to $\varepsilon P \geq \omega/(1-\omega)$. If $\varepsilon P < \omega/(1-\omega)$, then $\gamma(q^{OPT}) = \gamma(\omega) < \omega$, contradicting the presumption that $\gamma(q^{OPT}) \geq \omega$.

Consider next the case when $\gamma(q^{OPT}) < \omega$. Then $k(z, q^{OPT})$ is decreasing in $z$ for $z \in [\omega, 1]$, implying that $z^{OPT} = \omega$. To find $q^{OPT}$ we note that, under Assumption 1, it holds that $k(\omega, q)$ is decreasing in $q$ for $q \in [\omega, 1]$, implying that $q^{OPT} = \omega$.

We finally note that $\gamma(q^{OPT}) = \gamma(\omega) < \omega$ is equivalent to $\varepsilon P < \omega/(1-\omega)$.

We have thus shown that for $\varepsilon P \geq \omega/(1-\omega)$ it holds that $q^{OPT} = \omega$ and $z^{OPT} = \gamma(\omega) = \overline{z}$; and for $\varepsilon P \geq \omega/(1-\omega)$ it holds that $q^{OPT} = z^{OPT} = \omega$. The results are thus identical to those in Proposition 5, with $q^{OPT} = \omega$. $\|$

*Proof of Proposition 2*: First note that Assumption 2 in the paper ($\omega > \rho$) and $q^{EQ} \geq \omega$ imply that $q^{EQ} > \rho$. It follows that Proposition 9 holds, with $q^{EQ}$ replacing $\omega$ and $\eta(q^{EQ})$ replacing $\varepsilon$. Thus, for a given $q^{EQ}$, Proposition 9 tells us that $z^{EQ}$ is given by $J(z^{EQ}, q^{EQ}, P) = 0$ if $\eta(q^{EQ})P \geq q^{EQ}/(1-q^{EQ})$, and $z^{EQ} = q^{EQ}$ if $\eta(q^{EQ})P \leq q^{EQ}/(1-q^{EQ})$; note from (24) in the paper, and (19) in these notes, that $J(z, \omega, P) = F(z, P)$. It remains to be shown that $q^{EQ} = \omega$.

First note that $K(z, \widetilde{z}, q, \widetilde{q})$ is decreasing in $q$ for $z \geq \beta(\widetilde{z}, q, \widetilde{q})$, and $K(z, \widetilde{z}, q, \widetilde{q}) = 0$ for $z \leq \alpha(\widetilde{z}, q, \widetilde{q})$. It follows that $K(z, \widetilde{z}, q, \widetilde{q}) = \lambda(z, \widetilde{z}, q, \widetilde{q})A^{\rho}P^{1-\rho}$ on the relevant interval. To find $q^{EQ}$, we thus first differentiate $\lambda(z, \widetilde{z}, q, \widetilde{q})$ in (13) with respect to $q$, taking as given $\widetilde{q}$, $z$, and $\widetilde{z}$. We then evaluate this derivative at $\widetilde{z} = z$ and $\widetilde{q} = q$. If the derivative is negative for all $q \in [\omega, 1)$, and $z \in [q, 1)$, then the incumbent King maximizes $K(z, \widetilde{z}, q, \widetilde{q})$ by setting $q$ at its lowest permissible level, $\omega$. [Note that we can disregard $q = 1$ and $z = 1$, since $K(1, \widetilde{z}, q, \widetilde{q}) = K(z, \widetilde{z}, 1, \widetilde{q}) = 0$; see (16).] It must then hold that $q^{EQ} = \omega$.

Some algebra shows that

$$\frac{\partial \lambda(z, \widetilde{z}, q, \widetilde{q})}{\partial q} = (1-z)^{1-\rho}(1-q)^{\rho-1} \times$$

$$\left\{ \rho \left[ \eta(q)P - 1 - \left(\frac{q}{1-\widetilde{q}}\right)^{\rho} \left(\frac{z}{1-\widetilde{z}}\right)^{1-\rho} \right] + \rho \left(\frac{q}{1-\widetilde{q}}\right)^{\rho} \left(\frac{z}{1-\widetilde{z}}\right)^{1-\rho} \left(\frac{1-q}{q}\right) - \eta'(q)(1-q)P \right\}. \tag{A2}$$

The derivative in (A2) is negative if the expression in curly brackets is. Setting $\widetilde{z} = z$ and $\widetilde{q} = q$, after rearranging and dividing by $\eta(q)P$, we see that the expression in curly brackets is negative if

$$\begin{aligned} \frac{\eta'(q)(1-q)P}{\eta(q)P} &= \left[\frac{\eta'(q)q}{\eta(q)}\right] \left(\frac{1-q}{q}\right) \\ &> \rho \left[ \left(\frac{\eta(q)P - 1}{\eta(q)P}\right) + \frac{1}{\eta(q)P} \left(\frac{q}{1-q}\right)^{\rho} \left(\frac{z}{1-z}\right)^{1-\rho} \left(\frac{1-2q}{q}\right) \right]. \end{aligned} \tag{A3}$$

The first term inside the square brackets on the right-hand side of (A3), i.e. $[\eta(q)P - 1]/\eta(q)P$, is less than one; the second term is negative, since $q \geq \omega > 1/2$ (recall Assumption 1 in the paper). Thus the right-hand side is strictly less than $\rho$, and Assumption 2 ensures that the inequality in (A3) holds. Thus, $q^{EQ} = \omega$. As argued above, it then follows that Proposition 2 holds. $\|$

# References

[1] Lagerlöf, N.-P., 2009, Pacifying monogamy, mimeo, York University.