Supplement to

Long-Run Trends in Human Body Mass

by
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Deriving linear fertility

This setting shows how a functional form where fertility is linear in body mass can be derived from fundamentals. Agents have two ways to procure food: gathering plants (or hunting small prey); and hunting (big) animals. The former activity requires no energy or bodily input but depends on the level of technology; the latter requires no technology but relies only on body mass and energy (effort) spent.

Income from food gathering equals $L_t^{\delta} A_t$, where $\delta \in (0,1)$, and (like in the paper) L_t is resources (or land) per agent and A_t is technology.

To model big-prey hunting, let $e_{i,t}$ denote (physical) effort spent in hunting in period t by an agent of body type i. Food procured in this activity equals $De_{i,t}^{\gamma} L_t^{\omega} B_i$, where $D > 0$, $\omega \in (0,1)$, $\gamma \in (0,1)$, and B_i (recall) is body mass. Hunting also consumes an amount of energy which is proportional to the body mass of the hunter, and given by $e_{i,t} B_i$.

On top of energy used for hunting, bodies require energy for maintenance, which is given by αB_i.

The total energy surplus is used for reproduction. This gives fertility, $n_{i,t}$, as the sum of food procured in hunting and gathering, minus energy required in hunting and for body maintenance. That is,

$$n_{i,t} = L_t^{\delta} A_t + De_{i,t}^{\gamma} L_t^{\omega} B_i - \left[\alpha + e_{i,t}\right] B_i. \tag{1}$$

Agents choose effort exerted in hunting, $e_{i,t}$, to maximize fertility in (1). This gives optimal $e_{i,t}$ as

$$e_{i,t} = \left(\gamma D L_t^{\omega}\right)^{\frac{1}{1-\gamma}}. \tag{2}$$

Substituting optimal $e_{i,t}$ back into the expression for $n_{i,t}$ in (1) and working the algebra, gives

$$\begin{aligned} n_{i,t} &= L_t^{\delta} A_t + D\left(\gamma D L_t^{\omega}\right)^{\frac{\gamma}{1-\gamma}} L_t^{\omega} B_i - \left[\alpha + \left(\gamma D L_t^{\omega}\right)^{\frac{1}{1-\gamma}}\right] B_i \\ &= L_t^{\delta} A_t + D^{\frac{1}{1-\gamma}} \gamma^{\frac{\gamma}{1-\gamma}} (1-\gamma) L_t^{\frac{\omega}{1-\gamma}} B_i - \alpha B_i. \end{aligned} \tag{3}$$

As in the base-line setting, let total resources be normalized to one; then resources per agent is $L_t = 1/P_t$. Then we can write fertility in (3) as

$$n_{i,t} = \frac{A_t}{P_t^{\delta}} + \frac{\beta B_i}{P_t^{\omega/(1-\gamma)}} - \alpha B_i, \tag{4}$$

where $\beta = D^{\frac{1}{1-\gamma}} \gamma^{\frac{\gamma}{1-\gamma}} (1-\gamma)$.

This indeed resembles the expression for fertility, $n_{i,t}$, in the paper. In particular, if $\delta = \omega/(1-\gamma)$ it is seen that (4) above boils down to Eq. (7) in the paper, with η replaced by δ. The intuition is exactly the same as in the existing setting: as population rises the value of a big body declines due to falling productivity of the body-intensive activity, hunting.

Note also from the expression for optimal hunting effort in (2) that rising population density (leading to lower L_t) induces agents to allocate less effort to (big-prey) hunting and rely more on gathering (or small-prey hunting).

The qualitative results are unchanged if $\delta < \omega/(1-\gamma)$, which would be the case e.g. if $\delta = \omega$. The only difference is that the equation for the ($\Delta P_t = 0$)-locus (in the technologically stagnant phase) changes to

$$B_t = P_t^{\frac{\omega}{1-\gamma}-\delta} \left(\frac{P_t^{\delta} - A_0}{\beta - \alpha P_t^{\omega/(1-\gamma)}} \right). \tag{5}$$

instead of Eq. (12) in the paper. As long as $\delta < \omega/(1-\gamma)$ the slope is positive, as in Figure 2 in the paper, and the configuration thus the same.

Endogenous natural resource base

In this model income is given by

$$Y_{i,t} = L_t^{\eta} \left[A + \beta B_i \right] = \left(\frac{X_t}{P_t} \right)^{\eta} \left[A + \beta B_i \right]. \tag{6}$$

(Note that technology, A, is treated as constant.)

The dynamic equation for X_t becomes:

$$X_{t+1} = X_t + r X_t \left[1 - \frac{X_t}{\overline{X}} \right] - X_t^{\eta} P_t^{1-\eta} \left[A + \beta B_t \right], \tag{7}$$

where $X_t^{\eta} P_t^{1-\eta} \left[A + \beta B_t \right]$ is the amount harvested.

Fertility is given by income minus subsistence: $n_{i,t} = Y_{i,t} - \alpha B_i$. Using (6) fertility thus becomes:

$$n_{i,t} = \left(\frac{X_t}{P_t} \right)^{\eta} A + B_i \left[\beta \left(\frac{X_t}{P_t} \right)^{\eta} - \alpha \right]. \tag{8}$$

This gives a dynamic equation for population:

$$P_{t+1} = X_t^{\eta} P_t^{1-\eta} \left[A + \beta B_t \right] - \alpha P_t B_t. \tag{9}$$

Treating technology as constant there are now three state variables: P_t, B_t, and X_t. The dynamics for X_t and P_t are given by (7) and (9), respectively. The dynamics of B_t is given by noting that big types have the reproductive advantage ($\partial n_{i,t}/\partial B_i > 0$), and B_t is increasing over time, when the resource-to-population ratio, X_t/P_t, exceeds the threshold $(\alpha/\beta)^{1/\eta}$, and vice versa when this ratio falls below the threshold.

Steady-state levels are denoted with a superscript $*$.

At any given B^*, steady-state resource and population levels are given by setting $X_{t+1} = X_t = X^*$ and $P_{t+1} = P_t = P^*$ in (7) and (9), and solving for X^* and P^*. From (7) we get

$$P^* = \left[\frac{A + \beta B^*}{1 + \alpha B^*}\right]^{\frac{1}{\eta}} X^*, \tag{10}$$

The resource-to-population ratio thus equals $\{(1+\alpha B^*)/(A+\beta B^*)\}^{1/\eta}$ in steady state. Next use (8) and recall that big (small) types dominate if $\partial n_{i,t}/\partial B_i > (<) 0$. If the resource-to-population ratio exceeds $(\alpha/\beta)^{1/\eta}$ the reproductive success is greater for the big type [since $\partial n_{i,t}/\partial B_i$ is then positive; see (8)]. This amounts to $\{(1 + \alpha B^*)/(A + \beta B^*)\}^{1/\eta} > (\alpha/\beta)^{1/\eta}$, or (after a little algebra) $A < \beta/\alpha$. Vice versa, if $A > \beta/\alpha$ the small type has higher fertility. This implies that steady-state average body size, B^*, is given by

$$B^* = \begin{cases} \overline{B} & \text{if } A < \beta/\alpha \\ \underline{B} & \text{if } A > \beta/\alpha \end{cases}, \tag{11}$$

(and B^* can be anything between $\underline{B}$ and $\overline{B}$ if $A = \beta/\alpha$).

Using (9) gives

$$P^* = \left[\frac{r}{A + \beta B^*}\right]^{\frac{1}{1-\eta}} \left[1 - \frac{X^*}{\overline{X}}\right]^{\frac{1}{1-\eta}} X^*. \tag{12}$$

where (recall) B^* is given by (11). Equalizing (10) and (12), and solving for X^*, we get:

$$X^* = \overline{X}\left[1 - \frac{[A + \beta B^*]^{\frac{1}{\eta}}}{r(1 + \alpha B^*)^{\frac{\eta}{1-\eta}}}\right], \tag{13}$$

which is decreasing in A, holding fixed B^*. Using (10) and (13), we get an expression for P^*, not involving X^*:

$$P^* = \overline{X}\left(1 - \frac{[A + \beta B^*]^{\frac{1}{\eta}}}{r[1 + \alpha B^*]^{\frac{1-\eta}{\eta}}}\right)\left[\frac{A + \beta B^*}{1 + \alpha B^*}\right]^{\frac{1}{\eta}}. \tag{14}$$

From (14) it is seen that P^* is non-monotonic in A at any given level of B^*, reflecting two counteracting effects that technological progress has on steady-state population: higher technology means that more resources can be harvested, and thus a larger population at a given resource base; but it also means more resource depletion and thus fewer resources to harvest, meaning lower population.

Note also from (13) and (14) that $X^* = P^* = 0$ if $A = r^\eta(1 + \alpha B^*)^{1-\eta} - \beta B^*$.

CES production

In this setting technology and body mass are imperfect substitutes and food procurement is given by a function exhibiting constant elasticity of substitution:

$$Y_{i,t} = \frac{[A^\rho + \beta B_i^\rho]^{\frac{1}{\rho}}}{P_t^\eta}, \tag{15}$$

Technology, A, is treated as constant.

There is a continuum of body types, from $\underline{B}$ and $\overline{B}$. Using $n_{i,t} = Y_{i,t} - \alpha B_i$ and the implication that in steady state the economy is dominated by the body type for which $\partial n_{i,t}/\partial B_i = 0$, or $\partial Y_{i,t}/\partial B_i = \alpha$, we see that in steady state the average body size, B^*, is given by:

$$(P^*)^{-\eta}\left(\frac{1}{\rho}\right)\left[A^\rho + \beta\left(B^*\right)^\rho\right]^{\frac{1-\rho}{\rho}}\beta\rho\left(B^*\right)^{\rho-1} = \alpha, \tag{16}$$

where P^* is steady-state population size. Again using (15), and the fact that fertility of the type which dominates in steady state must equal unity, we can write:

$$1 = (P^*)^{-\eta}\left[A^\rho + \beta\left(B^*\right)^\rho\right]^{\frac{1}{\rho}} - \alpha B^*. \tag{17}$$

These two equations, (16) and (17), can be solved for B^* and P^*. The algebra is quite cumbersome; what follows is the route we have found easiest. First rewrite (16) as:

$$\left[A^\rho + \beta\left(B^*\right)^\rho\right]^{\frac{1}{\rho}} = \left(\frac{\alpha\left(P^*\right)^\eta}{\beta}\right)^{\frac{1}{1-\rho}} B^*, \tag{18}$$

or, raising both sides to ρ and rearranging:

$$\left(\frac{\alpha\left(P^*\right)^\eta}{\beta}\right)^{\frac{\rho}{1-\rho}} - \beta = \left(\frac{A}{B^*}\right)^\rho. \tag{19}$$

We then use (17) to write

$$\begin{aligned} 1 + \alpha B^* &= (P^*)^{-\eta}\left[A^\rho + \beta\left(B^*\right)^\rho\right]^{\frac{1}{\rho}} \\ &= (P^*)^{-\eta}\left(\tfrac{\alpha(P^*)^\eta}{\beta}\right)^{\frac{1}{1-\rho}} B^* \\ &= (P^*)^{\eta\left(\frac{1-(1-\rho)}{1-\rho}\right)}\left(\tfrac{\alpha}{\beta}\right)^{\frac{1-(1-\rho)}{1-\rho}}\left(\tfrac{\alpha}{\beta}\right) B^* \\ &= \left(\tfrac{\alpha(P^*)^\eta}{\beta}\right)^{\frac{\rho}{1-\rho}}\left(\tfrac{\alpha B^*}{\beta}\right) \end{aligned} \tag{20}$$

where the second equality uses (18). We can rewrite (20) as

$$\begin{aligned} 1 &= \left(\tfrac{\alpha B^*}{\beta}\right)\left[\left(\tfrac{\alpha(P^*)^\eta}{\beta}\right)^{\frac{\rho}{1-\rho}} - \beta\right] \\ &= \left(\tfrac{\alpha B^*}{\beta}\right)\left(\tfrac{A}{B^*}\right)^\rho \end{aligned} \tag{21}$$

where the second equality uses (19). This can be solved for B^* to give

$$B^* = \left[\frac{\beta}{\alpha A^\rho}\right]^{\frac{1}{1-\rho}}. \tag{22}$$

Substituting (22) back into (19) gives steady state population:

$$P^* = \left\{\left(\frac{\beta}{\alpha}\right)\left[\left(\frac{\alpha A}{\beta}\right)^{\frac{\rho}{1-\rho}} + \beta\right]^{\frac{1-\rho}{\rho}}\right\}^{\frac{1}{\eta}}. \tag{23}$$