PROBLEMS FOR ECON 2450

Preliminary, continuously updated

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1 Introduction

To be written.

A The IS-LM and Mundell-Fleming models

Problem A.1 Consider an IS-LM model where consumption, C, is given by the following consumption function:

$$C(Y - T) = a + b(Y - T). \tag{1}$$

where $a > 0$ and $0 < b < 1$.

(a) Graph C, as given by the function in (1), in a diagram with Y on the horizontal axis. Which is the lowest level of income (Y) that is consistent with non-negative consumption?

Now, let taxes be a function of income:

$$T = \tau Y, \tag{2}$$

where $0 < \tau < 1$.

(b) How does your answer under (a) change when taxes are given by (2)?

Recall that the IS curve is given by combinations of r and Y that satisfy

$$Y = C(Y - T) + I(r) + G. \tag{3}$$

(c) Use (1), (2), and (3) to find an expression for the slope of the IS curve (as drawn in a diagram with r on the vertical axis and Y on the horizontal axis). That is, find an expression for $\frac{dr}{dY}$ along the IS curve.

(d) We know that $I'(r) < 0$. What does your answer under (c) imply about the slope of the IS curve? (Is it positive or negative?)

Now let the investment function be given by

$$I(r) = \alpha r^{-\beta}, \tag{4}$$

where $\alpha > 0$ and $\beta > 0$.

(e) Say that we think that investment cannot be negative. Then why would the investment function in (4) be more reasonable than a linear one?

(f) Derive an expression for the IS curve when investment is given by (4). Your answer should be an equation with r on the left-hand side, and Y and exogenous parameters on the right-hand side.

(g) Draw the graph of the IS curve derived under (f). You will find that points on the IS curve cannot fall below some level of Y. (*Hint:* r goes to infinity as Y approaches that level from above.) Derive an expression for that minimum level, and explain intuitively why Y cannot fall below it.

Problem A.2 Consider the so-called Mundell-Fleming (or IS*-LM*) model, which describes a small open economy.

The exchange rate, denoted e, is the amount of foreign currency, say ¥ (Japanese yen), that one must pay to buy one unit of the domestic currency, say \$. A high e thus means that the domestic currency is expensive: foreigners pay a lot for the domestic currency, and domestic agents pay little for the foreign currency.

Let P_f be the price of foreign goods, and P_d the price of domestic goods. Also, let λ be the spending share on domestic goods in the consumption basket of the typical consumer in the domestic country, and $1-\lambda$ the spending share on foreign goods, where $0 < \lambda < 1$.

(a) To the domestic consumer the price of a foreign good in terms of the domestic currency is P_f/e. Explain why.

This means that the general price level in \$ in the domestic country, P, is given by

$$P = \lambda P_d + (1-\lambda)\frac{P_f}{e}. \tag{5}$$

The domestic interest rate is fixed and equal to the exogenous world interest rate, r^*. Like in the IS-LM model, demand for real money balances is given by $L(r^*, Y)$, which now depends on only one endogenous variable, Y. Recall that

$$\frac{\partial L(r^*, Y)}{\partial Y} > 0. \tag{6}$$

The equation for the LM* curve is $M/P = L(r^*, Y)$, or, together with (5):

$$\frac{M}{\lambda P_d + (1-\lambda)\frac{P_f}{e}} = L(r^*, Y). \tag{7}$$

(b) Determine the slope of the LM* curve in a diagram with e on the vertical axis and Y on the horizontal axis.

Let NX be net exports (i.e., exports minus imports); this is the same as the trade surplus. This is given by a net export function:

$$NX = NX(e). \tag{8}$$

Recall that to the domestic consumer a high e means a relatively low price of the foreign good compared to the domestic good, which should make exports low and imports high. Therefore, it is reasonable to assume that $NX'(e) < 0$.

The IS* curve is given by

$$Y = C(Y-T) + I^* + G + NX(e), \tag{9}$$

where $I^* = I(r^*)$ is the level of investment. Since r^* is exogenous we can treat I^* as exogenous too. Recall also that $0 < C'(Y - T) < 1$.

(c) Determine the slope of the IS* curve.

(d) Draw both the IS* and the LM* curves in the same diagram. Try to illustrate graphically the effects on e and Y from the following:
(i) an increase in G;
(ii) an increase in P_f;
(iii) an increase in M.

Problem A.3. The IS-rR Model

Consider a cousin of the IS-LM/IS*-LM* models, that we may call the IS-rR model. This model is meant to describe a small open economy that shares currency with the outside world, like e.g. Greece, which has the Euro as currency. Thus, in this model – different from the IS*-LM* model – there is no variable denoting the exchange rate.

The IS curve is very similar to what we discussed in class, and given by combinations of r and Y such that

$$Y = a + b(1 - \tau)Y + I(r) + G + NX,$$

where Y is GDP, G is government spending, NX is net exports, and $I(r)$ is investment, where r is the interest rate (in the home country). It is assumed that $I'(r) < 0$.

Consumption is given by $a + b(1 - \tau)Y$, where $a > 0$ and $0 < b < 1$; $Y(1 - \tau)$ is disposable income, where τ is the tax rate (and $0 < \tau < 1$). As in class, G, a, b, and τ are exogenous, and here NX is also exogenous (since there is no exchange rate that could effect net exports).

(a) Draw the IS curve in a diagram with r on the vertical axis and Y on the horizontal axis, and show how it shifts in response to an increase in G.

Here come the most important news. The home interest rate, r, may differ from the foreign interest rate, denoted r^*, by an amount given by a risk premium, which is a function of the budget deficit, $D = G - \tau Y$. The risk premium is denoted by the function $R(D)$, where $R'(D) > 0$. That is, the higher is the budget deficit, the higher is the risk premium. Intuitively, it is more expensive for governments to borrow money if they run large deficits, perhaps because indebted governments are more likely to default on their debts.

The rR-curve is now defined as combinations or r and Y such that

$$r - r^* = R(G - \tau Y),$$

where r^* is exogenous.

(b) Draw the rR curve in a diagram with r on the vertical axis and Y on the horizontal axis, and show how it shifts in response to an increase in G. (Hint: the slopes of the two curves do not need to have opposite signs.)

(c) Draw the IS and rR curves together, and show that if the curves intersect each other in a particular way, then an increase in G can cause a decrease in Y.

B Labor markets

Problem B.1 Consider a model of the labor market. There are $\overline{L}$ workers who are endowed with one unit of time each. For simplicity, we let time be indivisible: workers choose to either work, or stay at home. If they work, they earn a (real) wage, w; if they do not work they earn some benefit from the government, b. We assume that workers maximize their earnings, implying that total labor supply, L^S, is given by:

$$L^S = \begin{cases} \overline{L} & \text{if } w \geq b, \\ 0 & \text{if } w < b. \end{cases} \tag{10}$$

(a) Illustrate the supply function in (10) in a diagram with the price (here the real wage, w) on the vertical axis, and the quantity (L^S) on the horizontal axis.

Consider next the demand for labor, which is derived from the firm's profit maximization problem. There are many firms so each firm takes the wage rate as given. We normalize the price of the firms' output to one, and let the total output of a firm who hires L workers be AL^γ, where $A > 0$ and $0 < \gamma < 1$.

(b) Find an expression for a firm's profit, denoted π. (Recall that each worker costs w.)

(c) Use your answer under (b) to find the firms' profit-maximizing level of L.[1]

(d) If there are N firms, what is the aggregate demand for labor, L^D?

Finally, we can look at the labor market equilibrium, as given by setting $L^S = L^D$.

(e) What is the real wage in a labor market equilibrium without unemployment? How does the equilibrium wage rate depend on the size of the labor force, $\overline{L}$; productivity, A; and the number of firms, N?

(f) For what levels of A will there be full, or less than full, employment?

Problem B.2 Consider a similar setting as that in Problem B.1, but now assume that a union sets employment to maximize the expected income of each worker. More precisely, let the unemployment rate be u, which also denotes the probability that each worker faces of not finding a job; similarly, $1-u$ denotes the probability that a worker will find a job. Employed workers earn w and unemployed get a government subsidy of b. The expected income, here denoted E, is thus given by

$$E = ub + (1-u)w. \tag{11}$$

Let L be the number of workers hired by each of the N firms, and $\overline{L}$ the

[1] If you could not solve (b), let the profit expression be given by

$$\pi = AL^{1-\alpha} - wL,$$

for some $0 < \alpha < 1$.

total labor force. Then the unemployment rate is given by

$$u = \frac{\overline{L} - NL}{\overline{L}}. \tag{12}$$

The firms' profit maximization problem is the same as in Problem B.1. This can be seen to give the wage rate, w, as a function of the number of employed workers in each firm, L, as

$$w = \frac{\gamma A}{L^{1-\gamma}}. \tag{13}$$

(a) Use (11), (12) and (13) to derive an expression for E as a function of L and exogenous variables.

(b) The union chooses the level of employment, L, to maximize E. Find the union's optimal L. For what levels of A does the union choose L so that $NL < \overline{L}$ (less than full employment)? Is that level higher or lower compared to the perfect-competition case, as considered in Problem B.1?

(c) What is the wage rate for the employed workers? If there is unemployment, are employed workers better or worse off than unemployed workers? How does your answer contrast to the perfect-competition case?

(d) Write E as a function of w, instead of L, and show that the result is the same if we let the union set w instead of L.

Problem B.3 Let there be M identical workers, each with a unit time endowment. The workers care about consumption and leisure. There is no saving, so workers consume all their earnings. The utility of a worker who supplies an amount of labor l (where $0 \leq l \leq 1$) is given by

$$U = \lambda \ln(C) + (1 - \lambda) \ln(1 - l), \tag{14}$$

where C is consumption, l is labor supply, and λ is a parameter in the utility function ($0 < \lambda < 1$). A worker who does not work at all ($l = 0$) receives an unemployment benefit of $b > 0$, but does not receive any benefit is (s)he works ($l > 0$).

(a) Write a worker's budget constraint as a composite function. This should be an equation involving a "curly bracket," with C on the left-hand side and l, w, and b on the right-hand side. Note that either $l = 0$, or $l \in (0, 1]$.

(b) Find an expression for the utility a worker gets if not working at all ($l = 0$). Denote this utility $U^{\text{no work}}$. Your answer should be in terms of some, or all, of these variables: λ, b, and w.

(c) How much does an agent work if choosing to work at all?

(d) Find an expression for the maximum utility a worker gets if working, i.e., if (s)he sets $l \in (0, 1]$. Denote this utility U^{work}. Your answer should be in terms of some, or all, of these variables: λ, b, and w.

(e) What is the lowest level of w that would induce workers to work instead of living on benefits? Denote that level $\widehat{b}$, which is a function of b and λ. Is $\widehat{b}$ greater, or less, than b?

(f) Find an expression for aggregate labor supply, L^S. (Recall that there are M workers.) How does you answer compare to (10) in problem B.1?

Problem B.4 Consider a version of the model of the labor market discussed in B.1, but where the unemployment benefit, b, is endogenous and financed by taxes on wages, through an Employment Insurance system. The tax rate, τ, is exogenous ($0 < \tau < 1$). The before-tax wage rate is denoted w, so the after-tax wage is $(1 - \tau)w$.

There are $\overline{L}$ workers. Of these, L^D are employed and pay τw in tax each, and $\overline{L} - L^D$ are unemployed, each receiving a benefit b. The budget constraint for the Employment Insurance system can thus be written:

$$L^D \tau w = \left(\overline{L} - L^D\right) b, \tag{15}$$

which says that total expenditures on benefits, $\left(\overline{L} - L^D\right) b$, must equal total taxes collected from the employed, $L^D \tau w$.

The unemployment rate, u, is given by

$$u = \frac{\overline{L} - L^D}{\overline{L}}. \tag{16}$$

As before, labor demand is

$$L^D = N\left(\frac{\gamma A}{w}\right)^{\frac{1}{1-\gamma}}, \tag{17}$$

where the notation is the same as in B.1. No worker wants to work if the after-tax wage is lower than the benefit, $(1-\tau)w < b$, so labor supply is given by

$$L^S = \begin{cases} \overline{L} & \text{if } (1-\tau)w \geq b, \\ \\ 0 & \text{if } (1-\tau)w < b. \end{cases} \tag{18}$$

Throughout this exercise we assume that $u > 0$.[2]

(a) Draw the labor supply and labor demand curves in a diagram with w on the vertical axis and L^D and L^S on the horizontal axis. Draw the diagram so that there is less than full employment in equilibrium.

(b) Use (15) and (16) to write b as function of τ, u, and w.

(c) Find an expression for the (strictly positive) unemployment rate, u, as a function of exogenous parameters. (Hint: what is equilibrium w if there is less than full employment?)

(d) Find the before-tax wage rate, w, as a function of exogenous parameters. Is w increasing or decreasing in τ? What is the intuition?

(e) What about the after-tax wage rate? Is it increasing or decreasing in τ?

Problem B.5 This problem explores labor demand with nominal prices and wages. Let P be the dollar price of the output that the firm produces, which is taken to be the single consumption good in this economy. Also, let w^{NOM} be the nominal wage, i.e., the dollar wage that the firm pays to its workers. As in Problem B.1, output of a firm that hires L workers is AL^γ, where $A > 0$ and $0 < \gamma < 1$.

[2] It can be shown that b adjusts endogenously so that there is always some unemployment in this model, but you do not need to show that here.

(a) Write an expression for the firm's nominal profit (the profit in dollars) if it hires L workers. Denote this π^{NOM}.

(b) Find an expression for the optimal number of workers the firm hires. Your answer should be in terms of A, γ, P, and w^{NOM}.

(c) Let $w^{REAL} = w^{NOM}/P$. Explain in words why w^{REAL} is the real wage. That is, explain what w^{REAL} measures.

(d) Write the firm's demand for labor in terms of A, γ, and w^{REAL}.

(e) Holding w^{NOM} constant, what happens to the firm's demand for labor when P increases?

Problem B.6 Consider a labor market model with perfect competition again, as described in Problem B.1 and in the slides. There are N identical firms and $\overline{L}$ identical workers. A firm that hires L workers produces AL^{γ}, where $A > 0$ and $0 < \gamma < 1$. For simplicity, we here set the unemployment benefit to zero ($b = 0$), which means there is always full employment in equilibrium.

(a) The number of workers that each firm hires must equal $\overline{L}/N$ in equilibrium. Explain why in a few words.

(b) In this model, recall that the equilibrium wage rate equals $w = \gamma A(N/\overline{L})^{1-\gamma}$. Let $R = \overline{L}/N$. Write w in terms of R, A, and γ. Illustrate in a diagram with w on the vertical axis and R on the horizontal axis.

(c) What is each firm's profit (π) in equilibrium? Your answer should be an expression in terms of R, A, and γ. Illustrate in a diagram with π on the vertical axis and R on the horizontal axis. [Hint: recall that the profit of a firm that hires L workers equals $\pi = AL^{\gamma} - wL$ and use your answer under (b).]

(d) So far we have treated $\overline{L}$ and N as exogenous. Now suppose agents can choose to become workers or entrepreneurs (i.e., start a firm). What is the equilibrium level of R? How does it depend on firm productivity, A?

C Intertemporal models

Problem C.1 Consider a two-period model where the pre-tax income is Y_1 in the first period and Y_2 in the second. Consumption is C_1 in the first period, and C_2 in the second. Taxes and government spending are denoted T_1 and G_1 for the first period, and T_2 and G_2 for the second.

The government debt in the first period is the difference between its spending and its taxes:

$$D = G_1 - T_1, \tag{19}$$

where a negative D implies that the government accumulates assets. In the second period, the government must set taxes so that it pays for both its spending in that period, and the debt carried over from the previous, plus interest:

$$T_2 = G_2 + (1 + r)D. \tag{20}$$

Let S denote household savings. Then the first-period budget constraint of the households can be written

$$Y_1 - T_1 = C_1 + S, \tag{21}$$

and the second-period budget constraint becomes

$$Y_2 - T_2 + S(1 + r) = C_2. \tag{22}$$

(a) Use (21) and (22) to write an intertemporal budget constraint. This should be an equation with C_2 on the left-hand side, and Y_1, T_1, Y_2, T_2, r, and C_1 on the right-hand side. Draw the associated budget line in a diagram with C_2 on the vertical axis and C_1 on the horizontal. What is the slope? What is the vertical intercept?

(b) What is the intercept of the budget line derived under (a) if we impose zero government debt, $D = 0$?

(c) Use the expression you derived for the vertical intercept under (a) again, and then apply (19) and (20) to substitute for T_2 and D. Show that the budget line (in particular the vertical intercept) does not differ from your answer under (a). Why does it not matter whether debt is zero, positive, or negative?

Problem C.2 Consider a two-period model where there are no taxes, and no government spending. Agents earn an income Y in the first period, and nothing in the second. Consumption is C_1 in the first period, and C_2 in the second. The budget constraints can be written

$$C_1 = Y - S, \tag{23}$$

and

$$C_2 = S(1 + r), \tag{24}$$

where S is saving and r the interest rate.

Agents aim to maximize this utility function:

$$U = u(C_1) + \beta u(C_2), \tag{25}$$

where $\beta > 0$. That is, $u(C)$ is the utility an agent derives from consuming C. The larger is the parameter β, the higher weight the agent puts on consumption in the second period; usually we assume that agents discount the future, meaning that $\beta < 1$.

One common and practical utility function to work with is logarithmic utility:

$$u(C) = \ln(C). \tag{26}$$

(a) Substitute (23) and (24) into (25) to write overall utility, U, as a function of S.

(b) If the period utility function is given by (26), find the optimal level of S, as a function of β and Y. What is the fraction of income saved?

(c) Find an expression for $u'(C_1)/u'(C_2)$ as a function of β and r, when S is chosen optimally. Note that $u(C)$ now need not take the functional form in (26).

(d) Now think of period one as some arbitrary period (e.g., a particular year) that we call t, and period two thus being $t+1$. Use your answer under (c) to find an equation for $u'(C_t)/u'(C_{t+1})$. This is called an *Euler equation.*

Now again let the functional form for $u(C)$ be the one in (26).

(e) Use your answer under (d) to shown that

$$C_{t+1} = \beta(1+r)C_t, \tag{27}$$

for all $t \geq 0$; this is a so-called difference equation for C_t.

(f) Given the level of consumption in some initial period, C_0, use (27) to find an expression for C_t, as a function of C_0, t, β, and r.

Problem C.3 Consider yet another version of the previous two-period model, where a representative agent earns an income Y_1 in the first period, and Y_2 in the second. Consumption is C_1 in the first period, and C_2 in the second. The budget constraints of the representative agent can be written

$$C_1 = Y_1 - S, \tag{28}$$

and

$$C_2 = Y_2 + S(1+r), \tag{29}$$

where S is saving and r the interest rate. The agent maximizes this utility function:

$$U = (1-\beta)\ln(C_1) + \beta\ln(C_2), \tag{30}$$

where $\beta \in (0,1)$.

(a) Solve the utility maximization problem to derive an expression for S as a function of Y_1, Y_2, r, and β. (Hint: this is exactly the same problem we did in the slides.)

Next we let second-period production (and labor income) be determined by how much a representative firm invests in capital, K. (For simplicity, the firm is active only in the second period, so we do not need any sub-index on

K.) The total amount of goods produced is given by AK^α, where $A > 0$ and $\alpha \in (0, 1)$. Profits for the firm, π, are given by the amount produced, AK^α, minus the payments made for capital. Payments for capital in turn equal the sum of the capital invested, K, and the interest (or rental price) that must be paid on that capital, rK. This gives the representative firm's profit as follows:

$$\pi = AK^\alpha - (1 + r)K, \tag{31}$$

(b) Write the first-order condition that determines the profit-maximizing K.

In richer models it can be shown (although it is not shown here) that labor income in the second period equals $1 - \alpha$ (known as the labor share) times total output produced, AK^α. That is:

$$Y_2 = (1 - \alpha)AK^\alpha. \tag{32}$$

(c) Use (32) and your answer under (b) to write the present-value of second-period labor income, $Y_2/(1 + r)$, in terms of K and α.

(d) Use your answers under (a) and (c) to write S in terms of β, Y_1, K, and α.

(e) This is a closed economy so all of the capital that the representative firm invests, K, must be made up of savings by the representative agent, S. That is, $S = K$. Use this to write K in terms of β, α, A, and Y_1. How does K change is response to a higher β?

(f) Let the (net) growth rate in labor income between the two periods be denoted g, i.e., $g = (Y_2 - Y_1)/Y_1$. Find an expression for the gross growth rate, $1 + g$, in terms of first-period income, Y_1, and the exogenous parameters β, α, and A. Show that the growth rate is decreasing in first-period income. This phenomenon – that rich economies tend to grow slower than poor, and that poor economies thus tend to catch up with the rich – is known as *convergence*.

(g) Find an expression for $1 + r$ in terms of β, α, A, and Y_1.

(h) Show that consumption in the second period equals total output in a capital market equilibrium (i.e., when $S = K$). In other words, show that $C_2 = AK^\alpha$.

Problem C.4 Consider a version of the two-period model that describes a small open economy. Free capital mobility implies that the interest rate, r, equals that of the outside world, denoted r^*, which is here treated as exogenous. The representative firm's profit equals $\pi = AK^\alpha - (1 + r^*)K$.

(a) Denote the profit maximizing level of K by K^*. Write K^* in terms of r^*, A, and α.

(b) Second period labor income equals $Y_2 = (1 - \alpha)AK^\alpha$. Let Y_2^* denote the level of second period labor income when firms choose investment to maximize profits, i.e., when $K = K^*$. Show that

$$\frac{Y_2^*}{1 + r^*} = \left(\frac{1 - \alpha}{\alpha}\right) K^*. \tag{33}$$

(c) In this model K^* is the same as first period investment and Y_1 is the same as GDP in the first period. We assume away government consumption (setting $G = 0$ in both periods), so the National Accounts identity tells us that

$$Y_1 = C_1 + K^* + NX_1. \tag{34}$$

First period saving by the representative agent is given by $S = Y_1 - C_1$. Show that the first-period trade balance, NX_1, can be written as

$$NX_1 = S - K^*. \tag{35}$$

(d) We assume log utility with weight β on second-period consumption. With $Y_2^*/(1 + r^*)$ being the present value of second period labor income, recall that saving of the representative agent is given by

$$S = \beta Y_1 - (1 - \beta)\frac{Y_2^*}{1 + r^*}. \tag{36}$$

Use (36), (33) and your answer under (a) to derive an expression for the first period trade balance, NX_1, as a function of β, Y_1, r^*, A, and α. What does your answer say about the effect of an increase in productivity, A, on the first period trade balance?

(e) GDP in the second period equals $A\,(K^*)^\alpha$, which is just total output. There is no investment in the second period, and we have set government consumption to zero (in both periods). Therefore, the National Accounts identity becomes

$$A\,(K^*)^\alpha = C_2 + NX_2. \tag{37}$$

Setting $Y_2 = Y_2^*$, the second-period budget constraint for the agents states that

$$C_2 = Y_2^* + (1 + r^*)S \tag{38}$$

Show that $NX_2 = -(1 + r^*)NX_1$.[3] This means that if $NX_1 > 0$, then $NX_2 < 0$, and vice versa. Explain why.

D Monetary policy, rational expectations, and dynamic consistency

Problem D.1 Consider this aggregate demand function:

$$y = y^* + m - p, \tag{39}$$

where y is output, y^* is the long-run equilibrium level of output, m is money supply, and p the general price level, all in logarithms. Money supply, m, is a stochastic variable, with expected value $E(m)$, and it will be seen that y is also stochastic.

There is a large number of identical firms. Firm i sets the price of its output, p_i, before knowing the realization of m, as a function of the general price level and what the firm expects money supply to be. For the moment,

[3] Hint: show that

$$A\,(K^*)^\alpha = \frac{(1 + r^*)K^*}{\alpha}.$$

let the firms expect money supply to be $\widetilde{E}(m)$, where $\widetilde{E}(m)$ is the expectation under some "subjective" probability distribution, as described below. That is, $\widetilde{E}(m)$ may, or may not, be equal to $E(m)$. Firm i then sets its price according to:

$$p_i = ap + (1-a)\widetilde{E}(m), \tag{40}$$

where $0 < a < 1$. The general price level, p, is determined by the equilibrium condition:

$$p_i = p. \tag{41}$$

The timing of events is as follows:

- First, firms set prices, and the general price level is realized.
- Then "nature" selects a Central Banker (CB), who can be either "tough" or "soft." The (objective) probability that a tough CB is selected equals q, and the probability that a soft CB is selected is $1-q$, where $0 < q < 1$.
- If the CB is soft, he sets $m = \overline{m}$, and if he is tough he sets $m = \underline{m}$, where $\underline{m} < \overline{m}$.
- After m has been realized, y is determined according to (39) above.

(a) Find what the general price level, p, is as a function of $\widetilde{E}(m)$.

Assume that firms believe that the CB is going to be tough with probability $\widetilde{q}$, and soft with probability $1 - \widetilde{q}$, where $0 < \widetilde{q} < 1$. We call $\widetilde{q}$ the subjective probability, and q the objective probability. For the moment, we assume $\widetilde{q}$ need not equal q, i.e., firms may have *irrational expectations*. However, firms still know that a soft CB sets $m = \overline{m}$, and a tough CB sets $m = \underline{m}$.

(b) Find an expression for $\widetilde{E}(m)$ in terms of $\widetilde{q}$, $\underline{m}$, and $\overline{m}$.

(c) Let $\overline{y}$ be the level of y that is realized if the CB is soft, and let $\underline{y}$ be the level that is realized if he is tough. Find expressions for $\overline{y}$ and $\underline{y}$ in terms of $\widetilde{q}$, $\underline{m}$, and $\overline{m}$.

(d) Find $E(y)$ in terms of y^*, q, $\widetilde{q}$, $\underline{m}$, and $\overline{m}$. Note that $E(y)$ is calculated using the objective probability, q.

(e) Use your answer under (d) to see what happens if we now impose *rational expectations* ($\widetilde{q} = q$). Can $E(y)$ differ from y^*?

Problem D.2 Consider a model of dynamic consistency of monetary policy. The policymaker (a central banker, CB) minimizes a loss function, which depends on output, y, and inflation, π:

$$L = \frac{1}{2}\left(y - \widetilde{y}\right)^2 + \frac{\alpha}{2}\left(\pi - \widetilde{\pi}\right)^2, \tag{42}$$

where $\alpha > 0$, $\widetilde{y} > 0$, and $\widetilde{\pi} > 0$. We can interpret $\widetilde{y}$ and $\widetilde{\pi}$ the CB's most desired levels of output and inflation.

The so-called Phillips curve is similar to an aggregate demand curve, but with (log) money supply replaced by inflation. It postulates a relationship between output, y; actual inflation, π; and expected inflation, π^e. Following the notation used in the aggregate demand function earlier we write this as:

$$y = y^* + \phi(\pi - \pi^e), \tag{43}$$

where y^* is some equilibrium level of y. It is assumed that $y^* < \widetilde{y}$.

There are two different sets of assumptions about the timing of events under which we can solve this model: *discretion* and *commitment*. Under commitment the timing is as follows:

- The CB chooses π.
- The public forms expectations (i.e., π^e is set).
- Output is determined.

Under discretion the timing is as follows:

- The public forms expectations (π^e is set).
- The CB chooses π.

- Output is determined.

(a) Solve for the equilibrium levels of y and π under commitment.

(b) Solve for the equilibrium levels of y and π under discretion.

(c) Using answers under (a) and (b), is the CB better off under commitment or discretion?

E Political economics

Problem E.1 Consider a model where two parties (or politicians) compete to win an election. The voters' opinions are distributed uniformly on the interval $[0, 1]$. We can think of a voter close to zero as being far to the left, and a voter close to one as being far to the right.

The two parties are denoted L (left) and R (right). Each party first chooses a *platform*, which is represented by a point on $[0, 1]$. We denote L's platform by a_L, and R's by a_R. Voters vote on the party whose platform is closest to their own opinion.

The two parties also have preferred positions on $[0, 1]$, which could be interpreted as the opinions held by the party's own activists. Denote L's preferred position by $\widehat{a}_L$, and R's by $\widehat{a}_R$, where $0 < \widehat{a}_L < \widehat{a}_R < 1$. Choosing a platform which deviates from the preferred position can help win the election, but comes at some cost.

Let L's share of the vote be S_L, and R's share S_R.

(a) Recall that voters are uniformly distributed on $[0, 1]$ and vote on the party whose platform is closest to their own opinion. Use this information and some logical reasoning to show that:

$$\begin{aligned} S_L &= \tfrac{a_L + a_R}{2}, \\ S_R &= 1 - \tfrac{a_L + a_R}{2}. \end{aligned} \tag{44}$$

(You may assume that the parties make their choices to that $a_L < a_R$ always holds.)

The objective functions of L and R are given by

$$\begin{aligned} W_L &= \alpha S_L - \tfrac{1}{2}(a_L - \widehat{a}_L)^2, \\ W_R &= \alpha S_R - \tfrac{1}{2}(a_R - \widehat{a}_R)^2, \end{aligned} \tag{45}$$

where $0 < \alpha < \widehat{a}_R - \widehat{a}_L$.

Let the optimal platforms for L and R be denoted a_L^* and a_R^*, respectively. That is, a_L^* is the level of a_L that maximizes the objective function W_L in (45), subject to the expression for S_L in (44), and similarly for a_R^*.

(b) Find a_L^* and a_R^* in terms of $\widehat{a}_L$, $\widehat{a}_R$, and α. How do the parties choose their platforms if they do not care about the share of the vote?

(c) Show that $\alpha < \widehat{a}_R - \widehat{a}_L$ ensures that $a_L^* < a_R^*$.

(d) If $\widehat{a}_R + \widehat{a}_L > 1$, which party takes the larger share of the votes? What if $\widehat{a}_R + \widehat{a}_L < 1$?

Solutions

Problem A.1

(a) The graph is a straight line with slope b and vertical intercept $a - bT$. For $C \geq 0$ to hold requires

$$C(Y - T) = a + b(Y - T) \geq 0, \tag{46}$$

or

$$Y \geq T - \frac{a}{b}. \tag{47}$$

(b) From (1) and (2) and we get

$$\begin{aligned} C &= C(Y - \tau Y) \\ &= a + b(1 - \tau)Y. \end{aligned} \tag{48}$$

Since a, b, and Y are all non-negative, and $\tau < 1$ we see that $C \geq 0$ always holds.

(c) The slope of the IS curve becomes

$$\frac{dr}{dY} = \frac{\overbrace{1 - b(1 - \tau)}^{(+)}}{\underbrace{I'(r)}_{(-)}} < 0. \tag{49}$$

(d) As seen from (49) the slope is negative, because $I'(r) < 0$ and $b(1 - \tau) < 1$. (Recall that $b < 1$ and $0 < \tau < 1$.)

(e) The investment function in (4) implies that I cannot be negative for any r. Put another way, the *range* of $I(r) = \alpha r^{-\beta}$ does not include negative numbers.

(f) The expression for the IS curve becomes:

$$r = \left[\frac{\alpha}{Y[1 - b(1 - \tau)] - (a + G)} \right]^{\frac{1}{\beta}}. \tag{50}$$

(g) We see that r goes to infinity as the denominator in (50) goes to zero, which amounts to Y going to something we may call $\widetilde{Y}$, which is given by

$$\widetilde{Y} = \frac{a+G}{1-b(1-\tau)}. \tag{51}$$

Intuitively, Y cannot fall below $\widetilde{Y}$ because none of the components of GDP (C, I, and G) can be negative. If $I = 0$, they sum up to $C + I + G = a + b(1-\tau)Y + 0 + G$, which is the lowest value Y can take. Setting $Y \geq a + b(1-\tau)Y + G$, gives $Y \geq \frac{a+G}{1-b(1-\tau)} = \widetilde{Y}$.

Problem A.2

(a) Recall that e is the number of (say) ¥ that you pay per \$. Thus, $1/e$ is the number of \$ you pay per ¥. Recall also that P_f is the price of a foreign good in ¥. Thus, the number of \$ you must have to buy the foreign good is $P_f \times \frac{1}{e}$. In other words, $P_f \times \frac{1}{e}$ is the price of the foreign good in \$.

(b) Define the left-hand side of (7) as

$$\phi(e) = \frac{M}{\lambda P_d + (1-\lambda)\frac{P_f}{e}} = \frac{eM}{e\lambda P_d + (1-\lambda)P_f}. \tag{52}$$

It can then be seen that

$$\phi'(e) = \frac{M(1-\lambda)P_f}{[e\lambda P_d + (1-\lambda)P_f]^2} > 0. \tag{53}$$

We can write (7) as

$$\phi(e) = L(r^*, Y). \tag{54}$$

Differentiating with respect to Y (treating e as function of Y) gives

$$\frac{de}{dY} = \frac{\overbrace{\frac{\partial L(r^*, Y)}{\partial Y}}^{(+)}}{\underbrace{\phi'(e)}_{(+)}} > 0, \tag{55}$$

where we have used (6) and (53). We can also solve for e from (7) to get (57) below; this gives the same result, $\frac{de}{dY} > 0$.

(c) From (9) we get

$$\frac{de}{dY} = \frac{\overbrace{1 - C'(Y-T)}^{(+)}}{\underbrace{NX'(e)}_{(-)}} < 0. \tag{56}$$

(d)

(i) An increase in G shifts up the IS* curve at any given level of Y, which leads to higher e and Y.

To answer (ii) and (iii) we can solve (7) for e. This gives us an explicit expression for the LM* curve:

$$e = \frac{(1-\lambda)P_f L(r^*, Y)}{M - \lambda P_d L(r^*, Y)}. \tag{57}$$

For (ii) we see from (57) that an increase in P_f shifts up the LM* curve at any given level of Y. In the IS*-LM* diagram this leads to higher e and lower Y.

For (iii) we see that an increase in M shifts down the LM* curve at any given level of Y. In the IS*-LM* diagram this leads to lower e and higher Y.

Problem A.3

(a) The IS curve has negative slope and shifts out (or up) in response to an increase in G.

(b) The rR curve has negative slope and shifts out (or up) in response to an increase in G.

(c) Both shift out (or up) when G increases. Let IS be steeper than rR. Then, if rR shifts a lot and IS very little, the result can be a fall in Y.

Problem B.1

(a) The labor supply graph is given by two vertical lines: one from the origin to b on the vertical axis; and the other from b and up, from $\overline{L}$ on the horizontal axis.

(b) $\pi = AL^{\gamma} - wL$.

(c) $L = \left(\frac{\gamma A}{w}\right)^{1/(1-\gamma)}$; if you used the notation in the footnote, just replace γ by $1-\alpha$.

(d) $L^D = N\left(\frac{\gamma A}{w}\right)^{1/(1-\gamma)}$

(e) If $L^S = \overline{L}$, then $L^S = L^D$, using your answer under (d) above, will give

$$\overline{L} = N\left(\frac{\gamma A}{w}\right)^{1/(1-\gamma)}, \tag{58}$$

which can be solved for w to give:

$$w = \gamma A\left(\frac{N}{\overline{L}}\right)^{1-\gamma}. \tag{59}$$

The equilibrium wage rate in decreasing in the size of the labor force, $\overline{L}$, and increasing with productivity A, and the number of firms, N.

(f) For there to be full employment the wage rate derived under (e) must exceed b; see (10). That implies:

$$w = \gamma A\left(\frac{N}{\overline{L}}\right)^{1-\gamma} > b, \tag{60}$$

or, solving for A, we see that full employment occurs when

$$A > \frac{b}{\gamma}\left(\frac{\overline{L}}{N}\right)^{1-\gamma} = \widehat{A}, \tag{61}$$

and less than full employment of the above inequality in reversed. That is, $\widehat{A}$ is the level that A must exceed for unemployment to be zero.

Problem B.2

(a) Using (11), (12), and (13) gives

$$\begin{aligned} E &= ub + (1-u)w \\ &= \left[1 - \frac{NL}{\overline{L}}\right] b + \frac{NL}{\overline{L}}\frac{\gamma A}{L^{1-\gamma}} \\ &= b - \frac{NL}{\overline{L}} b + \frac{N}{\overline{L}}\gamma A L^{\gamma} \\ &= b + \left(\frac{N}{\overline{L}}\right)\left[\gamma A L^{\gamma} - bL\right] \end{aligned} \tag{62}$$

(b) The unconstrained choice of L is given by $L = \left(\frac{\gamma^2 A}{b}\right)^{1/(1-\gamma)}$. There is less than full employment if the union's unconstrained choice of L is such that $NL < \overline{L}$, i.e., if

$$NL = N\left(\frac{\gamma^2 A}{b}\right)^{1/(1-\gamma)} < \overline{L}, \tag{63}$$

which holds if

$$A < \frac{b}{\gamma^2}\left(\frac{\overline{L}}{N}\right)^{1-\gamma} = \left(\frac{1}{\gamma}\right)\widehat{A}, \tag{64}$$

and, vice versa, there is full employment is the inequality in (64) is reversed. Since $\gamma < 1$ (and thus $\frac{1}{\gamma} > 1$) the critical level that A must exceed for there to be zero unemployment is greater than the critical level derived under perfect comptition [see (61)]. In other words, there is unemployment for a broader range of values for A when unions set wages compared to the perfect-competition case.

(c) $w = \frac{b}{\gamma}$. Since $\gamma < 1$, the wage rate, w, is greater than the government benefit, b. That is, in a model where the union sets employment, workers who are employed are better off (earn more) than workers who are unemployed.

By contrast, under perfect competition, if there is unemployment, the wage rate equals b. That is, employed and unemployed workers have the same income.

(d) Find w that maximizes $E = b + \left(\frac{N}{\overline{L}}\right)\left(\frac{\gamma A}{w}\right)^{\frac{1}{1-\gamma}}(w-b)$; this gives $w = \frac{b}{\gamma}$. See class notes for details.

Problem B.3

(a) The budget constraint becomes:

$$C = \begin{cases} b & \text{if } l = 0, \\ wl & \text{if } l \in (0,1]. \end{cases} \tag{65}$$

(b)

$$U^{\text{no work}} = \lambda \ln(b) + (1-\lambda)\ln(1) = \lambda \ln(b) = \ln(b^{\lambda}). \tag{66}$$

(c) If working the worker sets $l = \lambda$.

(d) Setting $l = \lambda$ in (14) gives

$$\begin{aligned} U^{\text{work}} &= \lambda \ln(w\lambda) + (1-\lambda)\ln(1-\lambda) \\ &= \lambda \ln(w) + \lambda \ln(\lambda) + (1-\lambda)\ln(1-\lambda) \\ &= \ln\left\{w^{\lambda}\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right\}. \end{aligned} \tag{67}$$

(e) For a worker to choose to work it must hold that $U^{\text{work}} \geq U^{\text{no work}}$. Using (66) and (67) this gives:

$$\ln\left\{w^{\lambda}\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right\} \geq \ln(b^{\lambda}), \tag{68}$$

or

$$w \geq \frac{b}{\lambda(1-\lambda)^{\frac{1-\lambda}{\lambda}}} = \widehat{b} > b. \tag{69}$$

where the last inequality follows from $0 < \lambda < 1$, implying that the denominator is less than one.

(f) If $w \geq \widehat{b}$, each of the M workers supplies λ units of labor, so total labor supply becomes λM. If $w < \widehat{b}$, none of the M workers work, so total labor supply becomes 0. That is:

$$L^S = \begin{cases} \lambda M & \text{if } w \geq \widehat{b}, \\ 0 & \text{if } w < \widehat{b}. \end{cases} \tag{70}$$

This becomes identical to (10) if we denote λM by $\overline{L}$, and replace b in (10) by $\widehat{b}$. However, since $\widehat{b} > b$, the wage, w, must here exceed a higher threshold than the benefit, b, for workers to choose to work.

Problem B.4

(a) As in Problem B.1, but the labor supply graph now makes its horizontal jump at $w = b/(1-\tau)$.

(b) Dividing (15) by $\overline{L}$ gives $(1-u)\tau w = bu$, or:

$$b = \frac{(1-u)\tau w}{u}. \tag{71}$$

(c) If $u > 0$, then $w = b/(1-\tau)$. (This is seen by drawing L^S and L^D in a labor market diagram.) Using $w = b/(1-\tau)$ and the answer to (b) we get

$$\begin{aligned} b &= \frac{(1-u)\tau}{u} \frac{b}{(1-\tau)}, \\ u(1-\tau) &= (1-u)\tau, \\ u &= \tau. \end{aligned} \tag{72}$$

(d) Using the answer under (c), together with the expression for u in (16) and L^D in (17), we see that

$$1 - u = \frac{L^D}{\overline{L}} = \frac{N}{\overline{L}} \left(\frac{\gamma A}{w} \right)^{\frac{1}{1-\gamma}} = 1 - \tau, \tag{73}$$

which can be solved for w to give:

$$w = \gamma A \left[\frac{N}{\overline{L}(1-\tau)} \right]^{1-\gamma}. \tag{74}$$

As seen, w (the before-tax wage) is increasing in τ. Intuitively, higher taxes on those who are working, and more resources allocated as benefits to the unemployed, pulls more workers out of employment; note that the number of employed workers equals $\overline{L}(1-\tau)$. This in turn implies a higher marginal product of labor, and thus higher pre-tax wages.

(e) The after-tax wage rate equals

$$(1-\tau)w = (1-\tau)^{\gamma} \gamma A \left[\frac{N}{\overline{L}} \right]^{1-\gamma}, \tag{75}$$

which is decreasing in τ.

Problem B.5

(a) $\pi^{NOM} = PAL^{\gamma} - w^{NOM} L$.

(b) $L = \left(\frac{\gamma AP}{w^{NOM}} \right)^{\frac{1}{1-\gamma}}$.

(c) w^{REAL} measures how much the worker can buy of the consumption good (i.e., the same good that the firm produces).

(d) $L = \left(\frac{\gamma A}{w^{NOM}/P}\right)^{\frac{1}{1-\gamma}} = \left(\frac{\gamma A}{w^{REAL}}\right)^{\frac{1}{1-\gamma}}$.

(e) From (b) we see that an increase in P, given π^{NOM}, leads to an increase in the firms' labor demand.

Problem B.6

(a) With full employment, the total number of workers hired by all N firms must sum up to $\overline{L}$, implying that each firm hires $\overline{L}/N$ workers.

(b) Using $R = \overline{L}/N$ and the expression for w gives

$$w = \gamma A \left(\frac{N}{\overline{L}}\right)^{1-\gamma} = \gamma A \left(\frac{1}{R}\right)^{1-\gamma}, \tag{76}$$

which has negative slope in a diagram with w on the vertical axis and R on the horizontal axis.

(c) Each firm hires $R = \overline{L}/N$ workers. Substituting L for R in the expression for π and using (76) gives

$$\pi = AR^{\gamma} - \gamma A \left(\frac{1}{R}\right)^{1-\gamma} R = AR^{\gamma} - \gamma AR^{\gamma} = (1-\gamma)AR^{\gamma}, \tag{77}$$

which has positive slope in a diagram with π on the vertical axis and R on the horizontal axis.

(d) If $w > \pi$, more agents want to be workers, making R increase, and vice versa if $w < \pi$. The equilibrium level of R is therefore that at which $w = \pi$. Using (76) and (77) this gives

$$R = \frac{\gamma}{1-\gamma}, \tag{78}$$

which does not depend on A.

Problem C.1

(a) From (21) and (22) we get

$$C_2 = Y_2 - T_2 + S(1+r)$$

$$= Y_2 - T_2 + \underbrace{(Y_1 - T_1 - C_1)}_{S}(1+r) \tag{79}$$

$$= \underbrace{[Y_2 - T_2 + (Y_1 - T_1)(1+r)]}_{\text{vertical intercept}} - \underbrace{(1+r)}_{\text{slope}}C_1$$

(b) No government debt implies that taxes are equal to government spending in each period: $T_1 = G_1$ and $T_2 = G_2$. [To see this, set $D = 0$ in (19) and (20).] Setting $T_1 = G_1$ and $T_2 = G_2$ in (79) we get

$$C_2 = \underbrace{[Y_2 - G_2 + (Y_1 - G_1)(1+r)]}_{\text{vertical intercept}} - \underbrace{(1+r)}_{\text{slope}}C_1 \tag{80}$$

(c) From (79) we get

$$C_2 = [Y_2 - T_2 + (Y_1 - T_1)(1+r)] - (1+r)C_1$$

$$= Y_2 - \underbrace{\{G_2 + (1+r)D\}}_{T_2} + (Y_1 - T_1)(1+r) - (1+r)C_1$$

$$= Y_2 - \{G_2 + (1+r)\overbrace{(G_1 - T_1)}^{D}\} + (Y_1 - T_1)(1+r) - (1+r)C_1$$

$$= Y_2 - G_2 - (1+r)(G_1 - T_1) + (Y_1 - T_1)(1+r) - (1+r)C_1$$

$$= Y_2 - G_2 - (1+r)G_1 + (1+r)T_1 + (1+r)Y_1 - T_1(1+r) - (1+r)C_1$$

$$= Y_2 - G_2 - (1+r)G_1 + (1+r)Y_1 - (1+r)C_1$$

$$= \underbrace{[Y_2 - G_2 + (Y_1 - G_1)(1+r)]}_{\text{vertical intercept}} - \underbrace{(1+r)}_{\text{slope}}C_1 \tag{81}$$

which is identical to the answer under (b); see (80) above.

Debt does not matter because whatever the government borrows in the first period (instead of paying for with first-period taxes) it will have to pay back in the second period with higher taxes. The government in effect borrows on behalf of the agents. Since the government faces the same interest rate as agents when it borrows, the agents' budget constraints are not affected; they choose among the same consumption bundles as if there was no government debt. This is what is called Ricardian Equivalence.

Problem C.2

(a) $U = u(Y - S) + \beta u(S(1 + r))$.

(b) Using (26) the utility function becomes

$$U = \ln(Y - S) + \beta \ln(S(1 + r)). \tag{82}$$

The first-order condition for a maximum says that

$$\frac{\partial U}{\partial S} = \frac{-1}{Y - S} + \beta \frac{1 + r}{S(1 + r)} = 0, \tag{83}$$

or

$$\frac{1}{Y - S} = \frac{\beta}{S}. \tag{84}$$

Solving for S gives

$$S = \frac{\beta Y}{1 + \beta}. \tag{85}$$

The fraction of income (Y) that is saved thus equals $\beta/(1 + \beta)$.

(c) Here we maximize $U = u(Y - S) + \beta u(S(1+r))$ over S. The first-order condition says that:

$$\frac{\partial U}{\partial S} = -u'(\underbrace{Y - S}_{C_1}) + \beta u'(\underbrace{S(1 + r)}_{C_2})(1 + r) = 0 \tag{86}$$

or

$$\frac{u'(C_1)}{u'(C_2)} = \beta(1 + r). \tag{87}$$

(d) Replacing 1 by t, and 2 by $t+1$ in (87) gives:

$$\frac{u'(C_t)}{u'(C_{t+1})} = \beta(1+r) \tag{88}$$

(e) From (26) we see that $u'(C) = \frac{1}{C}$. Using (88), we get

$$\frac{u'(C_t)}{u'(C_{t+1})} = \frac{\frac{1}{C_t}}{\frac{1}{C_{t+1}}} = \frac{C_{t+1}}{C_t} = \beta(1+r). \tag{89}$$

which gives (27), i.e., $C_{t+1} = \beta(1+r)C_t$.

(f) Given C_0, (27) gives us C_1 as follows:

$$C_1 = \beta(1+r)C_0. \tag{90}$$

Using (27) and (90) we can then find the following expression for C_2:

$$C_2 = \beta(1+r)C_1 = \beta(1+r)\underbrace{[\beta(1+r)C_0]}_{C_1} = \{\beta(1+r)\}^2 C_0. \tag{91}$$

Using (27) and (91) we get C_3 the same way:

$$C_3 = \beta(1+r)C_2 = \beta(1+r)\underbrace{[\{\beta(1+r)\}^2 C_0]}_{C_2} = \{\beta(1+r)\}^3 C_0. \tag{92}$$

From (91) and (92) we should see the pattern now: the exponent on $\{\beta(1+r)\}$ increases by one unit for each period we jump forward. That is:

$$C_t = \{\beta(1+r)\}^t C_0, \tag{93}$$

for all periods $t \geq 0$.

Problem C.3

(a) See the slides for details:

$$S = \beta Y_1 - (1-\beta)\frac{Y_2}{1+r}. \tag{94}$$

(b) The first-order condotions states that:

$$\frac{\partial \pi}{\partial K} = \alpha A K^{\alpha-1} - (1+r) = 0. \tag{95}$$

(c) From (95) we get $(1+r) = \alpha A K^{\alpha-1}$. Together with (32) this gives:

$$\frac{Y_2}{1+r} = \frac{(1-\alpha)AK^{\alpha}}{\alpha A K^{\alpha-1}} = \left(\frac{1-\alpha}{\alpha}\right) K. \tag{96}$$

(d) Using (94) and (96) we get:

$$S = \beta Y_1 - (1-\beta)\frac{Y_2}{1+r} = \beta Y_1 - (1-\beta)\left(\frac{1-\alpha}{\alpha}\right) K. \tag{97}$$

(e) Setting S in (97) equal to K gives:

$$K\left[1 + (1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\right] = K\left[\frac{\alpha + (1-\beta)(1-\alpha)}{\alpha}\right] = \beta Y_1, \tag{98}$$

or

$$K = \left[\frac{\alpha\beta}{\alpha + (1-\beta)(1-\alpha)}\right] Y_1 = \left[\frac{\alpha\beta}{1-\beta(1-\alpha)}\right] Y_1, \tag{99}$$

where we note that $\alpha + (1-\beta)(1-\alpha) = 1 - \beta(1-\alpha)$. The denominator of the expression in square brackets in (99) is decreasing in β, and the numerator is increasing in β. Thus, K is increasing in β. Intuitively, more weight on second-period consumption leads to more saving in the first period and thus more capital accumulation.

(f) Using (99) and (32) we get

$$1 + g = \tfrac{Y_2}{Y_1} = (1-\alpha)A\left\{\overbrace{\left[\frac{\alpha\beta}{1-\beta(1-\alpha)}\right] Y_1}^{K}\right\}^{\alpha} \tfrac{1}{Y_1}$$

$$= (1-\alpha)A\left[\tfrac{\alpha\beta}{1-\beta(1-\alpha)}\right]^{\alpha} Y_1^{-(1-\alpha)}$$

which is decreasing in first-period income, Y_1, since $\alpha < 1$.

(g) Using (95) and (99) we get

$$\begin{aligned} 1+r &= \alpha A K^{\alpha-1} \\ &= \alpha A \left[\tfrac{\alpha\beta Y_1}{1-\beta(1-\alpha)}\right]^{\alpha-1} \\ &= \alpha A \left[\tfrac{1-\beta(1-\alpha)}{\alpha\beta}\right]^{1-\alpha} Y_1^{-(1-\alpha)}. \end{aligned}$$

(h) Set $S = K$, and use (95) to note that $(1+r)\,S = (1+r)K = \alpha A K^{\alpha}$. Then we see from (29) and (32) that

$$C_2 = Y_2 + (1+r)S = Y_2 + (1+r)K = (1-\alpha)AK^{\alpha} + \alpha A K^{\alpha} = AK^{\alpha}.$$

Problem C.4

(a) The first-order condition for a profit maximum is

$$\frac{\partial \pi}{\partial K} = \alpha A K^{\alpha-1} - (1+r^*) = 0. \tag{100}$$

The K that makes (100) hold is denoted K^*, so we can write

$$\alpha A\,(K^*)^{\alpha-1} = (1+r^*). \tag{101}$$

Solving for K^* we get:

$$K^* = \left(\frac{\alpha A}{1+r^*}\right)^{\frac{1}{1-\alpha}}. \tag{102}$$

(b) Setting $Y_2 = Y_2^*$ and $K = K^*$ gives

$$Y_2^* = (1-\alpha)A\,(K^*)^{\alpha}. \tag{103}$$

Diving by $1+r^*$ and using (101) gives

$$\frac{Y_2^*}{1+r^*} = \frac{(1-\alpha)A\,(K^*)^{\alpha}}{1+r^*} = \frac{(1-\alpha)A\,(K^*)^{\alpha}}{\alpha A\,(K^*)^{\alpha-1}} = \left(\frac{1-\alpha}{\alpha}\right)K^*. \tag{104}$$

(c) Use (34) and $S = Y_1 - C_1$ to see that

$$NX_1 = Y_1 - C_1 - K^* = S - K^*. \tag{105}$$

(d) Using (36), (104) and (105) we get

$$
\begin{aligned}
NX_1 &= S - K^* \\
&= \left[\beta Y_1 - (1-\beta)\tfrac{Y_2^*}{1+r^*}\right] - K^* \\
&= \left[\beta Y_1 - (1-\beta)\left(\tfrac{1-\alpha}{\alpha}\right)K^*\right] - K^* \\
&= \beta Y_1 - \left[\tfrac{(1-\beta)(1-\alpha)+\alpha}{\alpha}\right]K^*.
\end{aligned}
\tag{106}
$$

Finally, using (102) we get

$$
NX_1 = \beta Y_1 - \left[\frac{(1-\beta)(1-\alpha)+\alpha}{\alpha}\right]\left(\frac{\alpha A}{1+r^*}\right)^{\frac{1}{1-\alpha}}. \tag{107}
$$

We see from (107) that an increase in A lowers the trade balance, NX_1.
(e) From (37) and (38) follows that

$$
\begin{aligned}
NX_2 &= A(K^*)^\alpha - C_2 \\
&= A(K^*)^\alpha - [Y_2^* + (1+r^*)S] \\
&= A(K^*)^\alpha - (1+r^*)\left[\tfrac{Y_2^*}{1+r^*} + S\right].
\end{aligned}
\tag{108}
$$

Using (101) we see that

$$
A(K^*)^\alpha = \frac{(1+r^*)K^*}{\alpha}. \tag{109}
$$

Inserting (104) and (109) into (108) we get

$$
\begin{aligned}
NX_2 &= \tfrac{(1+r^*)K^*}{\alpha} - (1+r^*)\left[\left(\tfrac{1-\alpha}{\alpha}\right)K^* + S\right] \\
&= (1+r^*)\left\{\tfrac{K^*}{\alpha} - \left(\tfrac{1-\alpha}{\alpha}\right)K^* - S\right\} \\
&= (1+r^*)\left\{\tfrac{K^*}{\alpha} - \left(\tfrac{1}{\alpha} - 1\right)K^* - S\right\} \qquad (110) \\
&= (1+r^*)\left\{K^* - S\right\} \\
&= -(1+r^*)NX_1,
\end{aligned}
$$

where the very last equality uses (105).

To explain this equality, note that if a country runs a trade deficit in the first period ($NX_1 < 0$), it means that the country borrows from overseas. In the second period, it pays back (with interest) by running a trade surplus ($NX_2 > 0$).

Problem D.1

(a) Using (40) and (41) gives $p = \widetilde{E}(m)$.

(b) $\widetilde{E}(m) = \widetilde{q}\underline{m} + (1-\widetilde{q})\overline{m}$.

(c) Using (39) and the answers under (a) and (b) gives

$$y = y^* + m - \widetilde{E}(m) = y^* + m - [\widetilde{q}\underline{m} + (1-\widetilde{q})\overline{m}] \,. \qquad (111)$$

Setting $m = \overline{m}$ in (111) gives

$$\overline{y} = y^* + \overline{m} - [\widetilde{q}\underline{m} + (1-\widetilde{q})\overline{m}] = y^* + \widetilde{q}(\overline{m} - \underline{m}), \qquad (112)$$

and setting $m = \underline{m}$ in (111) gives

$$\underline{y} = y^* + \underline{m} - [\widetilde{q}\underline{m} + (1-\widetilde{q})\overline{m}] = y^* - (1-\widetilde{q})(\overline{m} - \underline{m}). \qquad (113)$$

(d) Using the answers under (c) gives

$$
\begin{aligned}
E(y) &= q\underline{y} + (1-q)\overline{y} \\
&= q\left[y^* - (1-\widetilde{q})(\overline{m} - \underline{m})\right] + (1-q)\left[y^* + \widetilde{q}(\overline{m} - \underline{m})\right] \\
&= y^* - \left[q(1-\widetilde{q}) - (1-q)\widetilde{q}\right](\overline{m} - \underline{m}) \\
&= y^* - \left[q - q\widetilde{q} - \widetilde{q} + q\widetilde{q}\right](\overline{m} - \underline{m}) \\
&= y^* + \left[\widetilde{q} - q\right](\overline{m} - \underline{m})
\end{aligned} \tag{114}
$$

(e) The answer under (d) shows immediately that if $\widetilde{q} = q$, then $E(y) = y^*$ always holds.

Problem D.2

(a) Under commitment we first impose $\pi = \pi^e$, which implies that $y = y^*$. This is substituted into the loss function:

$$L = \frac{1}{2}(y^* - \widetilde{y})^2 + \frac{\alpha}{2}\left(\pi - \widetilde{\pi}\right)^2. \tag{115}$$

We then choose π to minimize L in (115). The first-order condition says that $\alpha\left(\pi - \widetilde{\pi}\right) = 0$, or $\pi = \widetilde{\pi}$. Denoting the outcomes under commitment with a super-index C, we thus get: $y^C = y^*$ and $\pi^C = \widetilde{\pi}$.

(b) Under commitment the CB minimizes the loss function taking π^e as given. Substituting (43) into the loss function gives:

$$L = \frac{1}{2}[y^* + \phi(\pi - \pi^e) - \widetilde{y}]^2 + \frac{\alpha}{2}\left(\pi - \widetilde{\pi}\right)^2. \tag{116}$$

The first-order condition states that

$$[y^* + \phi(\pi - \pi^e) - \widetilde{y}]\phi + \alpha\left(\pi - \widetilde{\pi}\right) = 0. \tag{117}$$

The outcome under discretion is then given by setting $\pi = \pi^e$ in (117), which gives $\pi = \widetilde{\pi} + \frac{\phi}{\alpha}(\widetilde{y} - y^*)$. Note also that $\pi = \pi^e$ implies $y = y^*$. Denoting the outcomes under discretion with a super-index EQ, we then get: $y^{EQ} = y^*$ and $\pi^{EQ} = \widetilde{\pi} + \frac{\phi}{\alpha}(\widetilde{y} - y^*)$.

(c) Output is the same under commitment as under discretion. However, inflation is higher under discretion, and further away form the preferred level of inflation, $\widetilde{\pi}$. Thus the CB is worse off under discretion. To see this one can also use the answers under (a) and (b) to calculate the value of the loss function to see that the loss is bigger under discretion than under commitment.

Problem E.1

(a) To make sense of this you should draw the unit line and illustrate how a_L and a_R are positioned on it. We then see the following:

- L takes all votes to his left, which equals a_L; he also takes half of the votes on the interval between him and R, i.e., $[a_L, a_R]$, the length of which is $a_R - a_L$. This gives $S_L = a_L + (a_R - a_L)/2 = (a_L + a_R)/2$.
- R takes all votes to his right, which equals $1 - a_R$; he also takes half of the votes on the interval between him and L, the length of which we just noted is $a_R - a_L$. This gives $S_R = 1 - a_R + (a_R - a_L)/2 = 1 - (a_L + a_R)/2$.

Note that $S_R + S_L = 1$.

(b) The first-order conditions give:

$$\begin{aligned} a_L^* &= \widehat{a}_L + \tfrac{\alpha}{2} \\ a_R^* &= \widehat{a}_R - \tfrac{\alpha}{2} \end{aligned} \tag{118}$$

If L and R do not care about S_L and S_R, that means $\alpha = 0$ in (45), which gives $a_L^* = \widehat{a}_L$ and $a_R^* = \widehat{a}_R$. The parties simply pick their preferred positions.

(c) Using (118) we get

$$\begin{aligned} a_L^* - a_R^* &= \widehat{a}_L - \widehat{a}_R + \frac{\alpha}{2} - \left(-\frac{\alpha}{2}\right) \\ &= \alpha - [\widehat{a}_R - \widehat{a}_L] \end{aligned} \tag{119}$$

We then see from (119) that $\widehat{a}_R - \widehat{a}_L > \alpha$ implies that the right-hand side is negative, and thus $a_L^* < a_R^*$.

(d) The shares are given by (44), with a_L and a_R substituted for by a_L^* and a_R^* in (118). Denoting the shares S_L^* and S_R^* we can write

$$\begin{array}{c} S_L^* = \frac{a_L^*+a_R^*}{2} = \frac{\widehat{a}_L+\widehat{a}_R}{2}, \\ S_R^* = 1 - \frac{a_L^*+a_R^*}{2} = 1 - \frac{\widehat{a}_L+\widehat{a}_R}{2}. \end{array} \tag{120}$$

Thus, L takes more votes than R if $S_L^* > S_R^*$, which happens if

$$\frac{a_L^* + a_R^*}{2} = \frac{\widehat{a}_L + \widehat{a}_R}{2} > \frac{1}{2} \tag{121}$$

or $\widehat{a}_L + \widehat{a}_R > 1$. Vice versa, if $\widehat{a}_L + \widehat{a}_R < 1$, then R takes more votes the L.

Intuitively, the further to the right the parties' preferred positions are, the larger vote share goes to the party that is relatively far to the left. Vice versa, the further to the left the parties' preferred positions are, the larger vote share goes to the party to the right. You are more likely to win if you run against an relatively extreme opponent, and/or if your own preferred position is relatively moderate.