Midterm Exam – Econ 4020
19 November 2015
Department of Economics
York University

Instructions: Unless otherwise explicitly stated, you must show how you arrived at your answer to get full mark. However, where the answer is just one, or a few, words no further motivation is needed. Answers must be given on the answer sheets provided. Ask for extra answer sheets if needed. Do not fold the answer sheets or write on the back.

1. The Solow model [10 marks]

Consider a standard Solow model, where the capital stock per effective worker, k, evolves over time according to

$$\dot{k} = sf(k) - (n + g + \delta)k.$$

where $f(k)$ is the intensive-form production function, which is such that $f'(k) > 0$ and $f''(k) < 0$. Let $r = f'(k) - \delta$ be the real interest rate, where δ is the capital depreciation rate. Note that k and r are all dependent on time but we suppress the time argument, while s, n, g, and δ are strictly positive constants (and $0 < s < 1$).

We denote the steady state level of k by k^*, and the corresponding level of r by r^*.

(a) Let $c^* = (1 - s)f(k^*)$ be steady-state consumption per effective worker. Show that the golden-rule level of k^*, denoted k_{GR}, is such that $f'(k_{GR}) = n + g + \delta$. [4 marks]

(b) Assume that exogenous parameters are such that $k^* > k_{GR}$. Show what this implies about how the steady-state real interest rate, r^*, relates to the economy's growth rate, $n+g$. That is, show if $r^* > n + g$ or $r^* < n + g$. [4 marks]

(c) Now assume Cobb-Douglas production, $f(k) = k^\alpha$, where $0 < \alpha < 1$. Find an expression for s_{GR}, i.e., the level of s that makes $k^* = k_{GR}$. You may use what you showed under (a). [2 marks]

2. The Ramsey model [10 marks]

In a standard Ramsey model the dynamics of consumption per effective worker, c, is given by the so-called Euler equation:

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho - \theta g}{\theta},$$

where $f'(k) > 0$ and $f''(k) < 0$. (We assume that depreciation is zero.) Similarly, the dynamics of k is given by

$$\dot{k} = f(k) - c - (n + g)k.$$

Let

$$\alpha(k) = \frac{f'(k)k}{f(k)}.$$

Here k and c are dependent on time but we suppress the time argument, while ρ, n, g and θ are all strictly positive constants.

(a) Draw the ($\dot{c} = 0$)-locus in a phase diagram and explain how it is derived. Draw motion arrows to show how c changes over time at positions off the locus, and explain why. [4 marks]

(b) Show in a phase diagram how the ($\dot{c} = 0$)-locus shifts in response to an increase in θ. [3 marks]

(c) Find an expression for the consumption-income ratio in steady state, $c^*/f(k^*)$, as a function of $\alpha(k^*)$ and (some or all of) the exogenous parameters n, ρ, θ and g. [3 marks]

For 2 (b) you do not need to show how you arrived at your answer, just draw things correctly.

3. Endogenous Growth [10 marks]

Consider an endogenous growth model. Total output, Y, is produced with the production function

$$Y = A\left([1 - a_L]\, L\right)^{\rho},$$

where ρ is an exogenous parameter, such that $0 < \rho \leq 1$, A is the level of technology (the number of ideas), L is the total labor force, and $1 - a_L$ is the fraction workers employed in goods production. Here Y, L, and A depend on time but we suppress the time argument, while a_L and ρ are exogenous and do not depend on time. New ideas, $\dot{A}$, are produced with the production function

$$\dot{A} = B(a_L L)^{\gamma} A^{\theta}.$$

where θ, γ, and B are strictly positive, exogenous and constant over time. We assume that $\theta < 1$ and that L grows at the exogenous rate $n > 0$. Output per worker is denoted by $x = Y/L$.

(a) Find an expression for the steady-state growth rate of A (denoted g_A^*) in terms of the exogenous variables. [5 marks]

(b) Assume that $\gamma > (1 - \theta)(1 - \rho)$. Denote the steady-state growth rate of output per worker by g_x^*. Derive an expression for g_x^* in terms of exogenous variables and show that $g_x^* > 0$. [5 marks]

Answer sheet for Problem___Econ 4020 Midterm Exam 19 November 2015

Student Name: SID Number:

____

Write your answer(s) below:

Sketches of solutions to select problems

1. (a) $\dot{k} = 0$ gives $f(k^*) = (n + g + \delta)k^*$, and thus

$$c^* = (1 - s)f(k^*) = f(k^*) - (n + g + \delta)k^*.$$

Using that k^* is increasing in s (which can be shown in a diagram or through implicit differentiation) we see that c^* is maximized when

$$\frac{dc^*}{ds} = [f'(k^*) - (n + g + \delta)] \frac{dk^*}{ds} = 0,$$

which gives the condition asked for.

(b) $k^* > k_{GR}$ and $f''(k) < 0$ imply

$$f'(k^*) < f'(k_{GR}) = n + g + \delta$$

meaning $r^* = f'(k^*) - \delta < n + g$.

(c) Use $sf(k^*) = s(k^*)^\alpha = (n + g + \delta)k^*$, and $f'(k_{GR}) = \alpha(k_{GR})^{\alpha-1} = n + g + \delta$, and set $k_{GR} = k^*$. Solving for s gives $s_{GR} = \alpha$.

2 (a) $\dot{c} = 0$ when $k = k^*$, defined from $f'(k^*) = \rho + \theta g$. The locus is thus a vertical line in a diagram with c on the vertical axis and k on the horizontal. For $k < k^*$, it holds that $f'(k) > \rho + \theta g$, and thus $\dot{c} > 0$, and vice versa for $k > k^*$.

(b) The locus shifts to the left.

(c)

$$\begin{aligned}
\frac{c^*}{y^*} &= \frac{c^*}{f(k^*)} = \frac{f(k^*) - (n + g)k^*}{f(k^*)} \\
&= 1 - (n + g)\left(\frac{k^* f'(k^*)}{f(k^*)}\right)\frac{1}{f'(k^*)} \\
&= 1 - \frac{(n + g)\alpha(k^*)}{\rho + \theta g}
\end{aligned}$$

3 (a)

$$g_A^* = \frac{\gamma n}{1 - \theta}$$

(b)

$$\begin{aligned}
g_x^* &= g_A^* - (1 - \rho)n \\
&= \left[\frac{\gamma}{1 - \theta} - (1 - \rho)\right] n \\
&= \left[\frac{\gamma - (1 - \theta)(1 - \rho)}{1 - \theta}\right] n.
\end{aligned}$$