Midterm Exam – Econ 4020
20 November 2014
Department of Economics
York University

Note: unless otherwise explicitly stated, for full mark you must show how you arrived at your answer.

1. The Solow model [10 marks]

Consider the Solow model, where k evolves according to $\dot{k} = \phi(k)$, where

$$\phi(k) = sk^{\alpha} - (n + g + \delta)k.$$

The linearization of this equation around the steady-state level of k, denoted k^*, is given by $\dot{k} = \psi(k)$, where

$$\psi(k) = \phi(k^*) + \phi'(k^*)(k - k^*).$$

(a) Find an expression for $\psi(k)$ as the product of $(k - k^*)$ and something involving α, n, g, and δ. [5 marks]

(b) Find an expression for the "gap" at time t, given by $k(t) - k^*$. Your answer should be the product of two factors: (1) the initial gap, $k(0) - k^*$; and (2) something that involves time, t, and the parameters α, n, g, and δ. To get there you must solve the linear differential equation $\dot{k} = \psi(k)$. Hint: if $\dot{x} = ax$, then $x(t) = x(0)e^{at}$. [5 marks]

2. The Ramsey model [10 marks]

In a standard Ramsey model the dynamics of consumption per effective worker, c, is given by the so-called Euler equation:

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho - \theta g}{\theta},$$

where $f'(k)$ is the real interest rate, r. (We assume that depreciation is zero.) Similarly, the dynamics of k is given by

$$\dot{k} = f(k) - c - (n + g)k.$$

Both c and k are functions of time, which we suppress, and we let c^* and k^* denote their respective steady state levels.

(a) Find an expression for the steady state real interest rate, $r^* = f'(k^*)$, in terms of some, or all, of the exogenous parameters n, ρ, θ, and g. [2 marks]

For Problem 2 (b)-(e) below, you do not need to motivate anything, just draw everything correctly.

(b) Draw the ($\dot{k} = 0$)- and ($\dot{c} = 0$)-loci in a phase diagram, and indicate c^* and k^* on the relevant axes. [2 marks]

(c) In the same type of phase diagram as under (b), draw the saddle path leading to steady state. [2 marks]

(d) Suppose that the representative household has a finite time horizon, and cares about consumption only up to some point in time $T > 0$.[1] In the same type of phase diagram as under (b) and (c), draw the path that this economy follows. Assume that $k(0) < k^*$. Indicate where the economy is at time T. [2 marks]

(e) Under the same assumptions as in (d), draw the time path of k, in a diagram with t on the horizontal axis and k on the vertical. Also indicate k^* on the vertical axis. [2 marks]

3. Endogenous growth [10 marks]

Let K, L, and A be levels of capital, labor, and technology. K and A evolve over time according to

$$\begin{aligned} \dot{K} &= c_K K^\alpha (AL)^{1-\alpha}, \\ \dot{A} &= c_A K^\beta L^\gamma A^{1-\beta}. \end{aligned}$$

Here K and A are functions of time (but we suppress their time arguments) while L, c_K, c_A, α, β, and γ are constant and treated as exogenous.

(a) Let $g_K = \dot{K}/K$, and $g_A = \dot{A}/A$. Show that

$$\begin{aligned} g_K &= c_K L^{1-\alpha} \left(\frac{A}{K}\right)^{1-\alpha}, \\ g_A &= c_A L^\gamma \left(\frac{A}{K}\right)^{-\beta}. \end{aligned}$$

[3 marks]

(b) Let $R = A/K$. Write $\dot{R}/R$ as a function of R and things that are constant over time. Illustrate in a diagram with $\dot{R}/R$ on the vertical axis and R on the horizontal, and indicate the steady-state level of R. [3 marks]

(c) Find an expression for the steady-state level of R in terms of L, c_K, c_A, α, β, and γ. (This is the level of R on the balanced growth path, where $g_K = g_A = g$ is constant.) [4 marks]

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[1] This note does not provide any crucial information, but to be explicit about what a finite time horizon means here: if $C(t)$ is consumption per worker (or household member) at time t, and $L(t)$ is the size of the household at time t, then the household's utility is given by $U = \int_0^T e^{-\rho t} L(t) \frac{[C(t)]^{1-\theta}}{1-\theta} dt$, where T is (strictly positive and) finite.

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Note: do not write on the back of the answer sheet

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Write your answer(s) below:

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Write your answer(s) below:

Sketches of solutions

1 (a)

$$\begin{aligned}\psi(k) &= \phi(k^*) + \phi'(k^*)\,(k-k^*) \\ &= 0 + \left[s\alpha(k^*)^{\alpha-1} - (n+g+\delta)\right](k-k^*) \\ &= \left[s\alpha\left(\frac{n+g+\delta}{s}\right) - (n+g+\delta)\right](k-k^*) \\ &= -(1-\alpha)(n+g+\delta)(k-k^*),\end{aligned}$$

where the third equality comes from $k^* = [s/(n+g+\delta)]^{1/(1-\alpha)}$. Note that $\phi(k^*) = 0$ from the definition of k^* (i.e., $\dot{k} = 0$ at $k = k^*$).

(b) Let $\chi = k - k^*$ and $\lambda = (1-\alpha)(n+g+\delta)$, so that $\dot{k} = \psi(k) = -\lambda\,(k-k^*)$. Since k^* does not depend on time, it follows that

$$\dot{\chi} = \dot{k} = \psi(k) = -\lambda(k-k^*) = -\lambda\chi,$$

which has solution $\chi(t) = \chi(0)e^{-\lambda t}$, or

$$k(t) - k^* = [k(0) - k^*]\, e^{-(1-\alpha)(n+g+\delta)t}.$$

2. (a) Setting $\dot{c} = 0$ gives $r^* = f'(k^*) = \rho + \theta g$.
(b)-(e) See attached scanned hand-drawn diagrams.

3. (After the typos are corrected)
(a)

$$\begin{aligned}g_K &= \frac{\dot{K}}{K} = c_K K^{\alpha-1}(AL)^{1-\alpha} = c_K L^{1-\alpha}\left(\frac{A}{K}\right)^{1-\alpha}, \\ g_A &= \frac{\dot{A}}{A} = c_A K^{\beta} L^{\gamma} A^{-\beta} = c_A L^{\gamma}\left(\frac{A}{K}\right)^{-\beta}.\end{aligned}$$

(b) With $R = A/K$ it follows that

$$\begin{aligned}\frac{\dot{R}}{R} &= \frac{\dot{A}}{A} - \frac{\dot{K}}{K} = g_A - g_K \\ &= c_A L^{\gamma}\left(\frac{A}{K}\right)^{-\beta} - c_K L^{1-\alpha}\left(\frac{A}{K}\right)^{1-\alpha} \\ &= c_A L^{\gamma} R^{-\beta} - c_K L^{1-\alpha} R^{1-\alpha}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial(\frac{\dot{R}}{R})}{\partial R} &= \frac{\partial\left(c_A L^{\gamma} R^{-\beta} - c_K L^{1-\alpha} R^{1-\alpha}\right)}{\partial R} \\ &= -\underbrace{\beta c_A L^{\gamma} R^{-\beta-1}}_{(+)} - \underbrace{(1-\alpha) c_K L^{1-\alpha} R^{-\alpha}}_{(+)} < 0.\end{aligned}$$

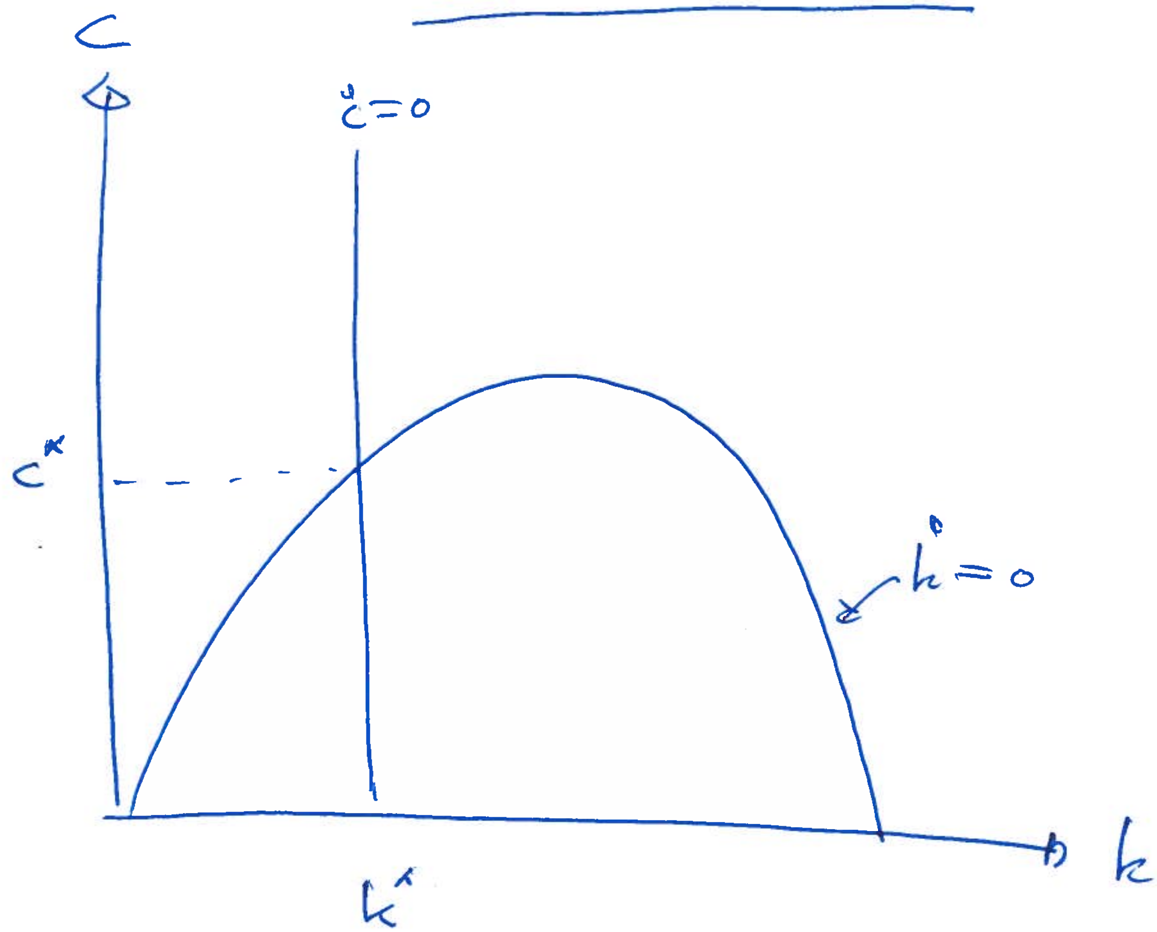
So when you draw $\overset{\bullet}{R}/R$ in a diagram with R on the horizontal axis, the curve should have negative slope. The steady state level of R is where this curve intersects zero, since this is where $\overset{\bullet}{R} = 0$, meaning R is constant.

(c) Setting $\overset{\bullet}{R} = c_A L^\gamma R^{-\beta} - c_K L^{1-\alpha} R^{1-\alpha} = 0$ (or $g_A = g_K$), and then solving for R gives the steady state level of R (call it R^*) as follows:

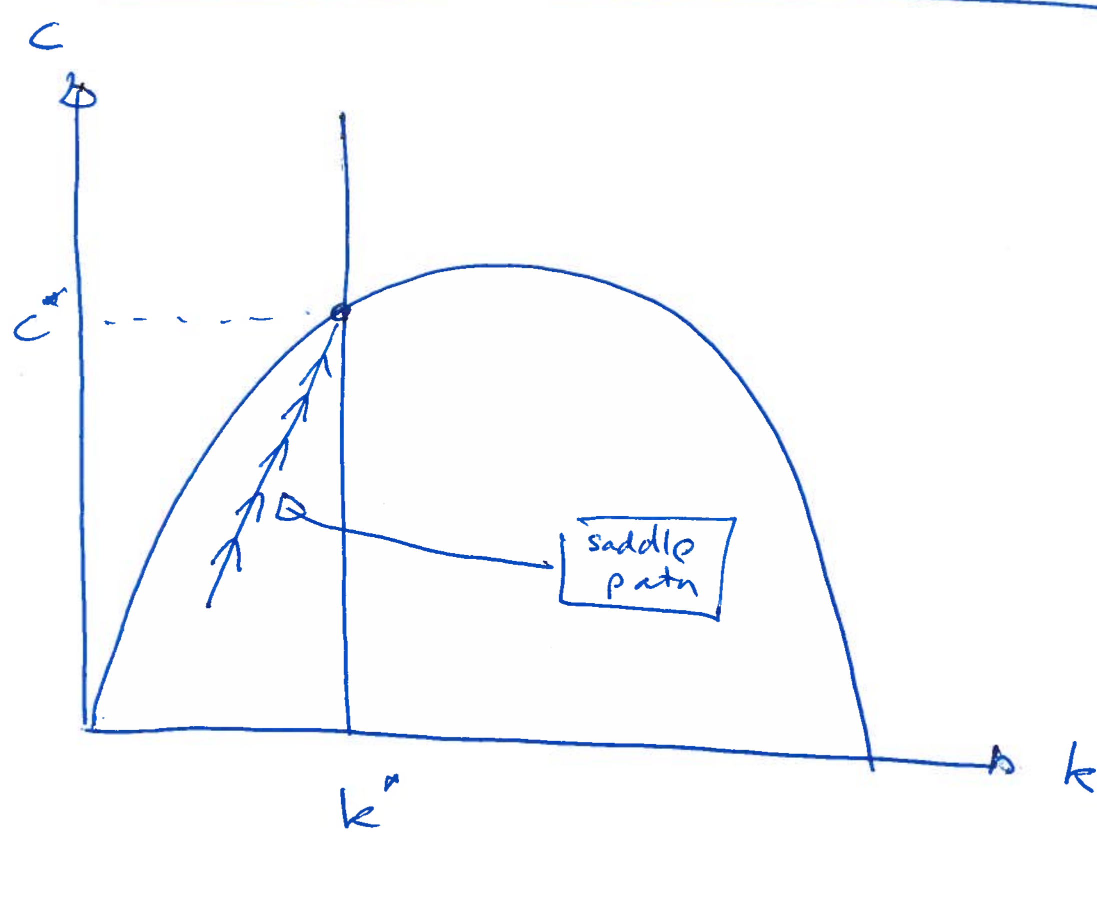
$$R^* = \left[\left(\frac{c_A}{c_K} \right) L^{\alpha+\gamma-1} \right]^{\frac{1}{1+\beta-\alpha}}.$$

2 (b) – (e)

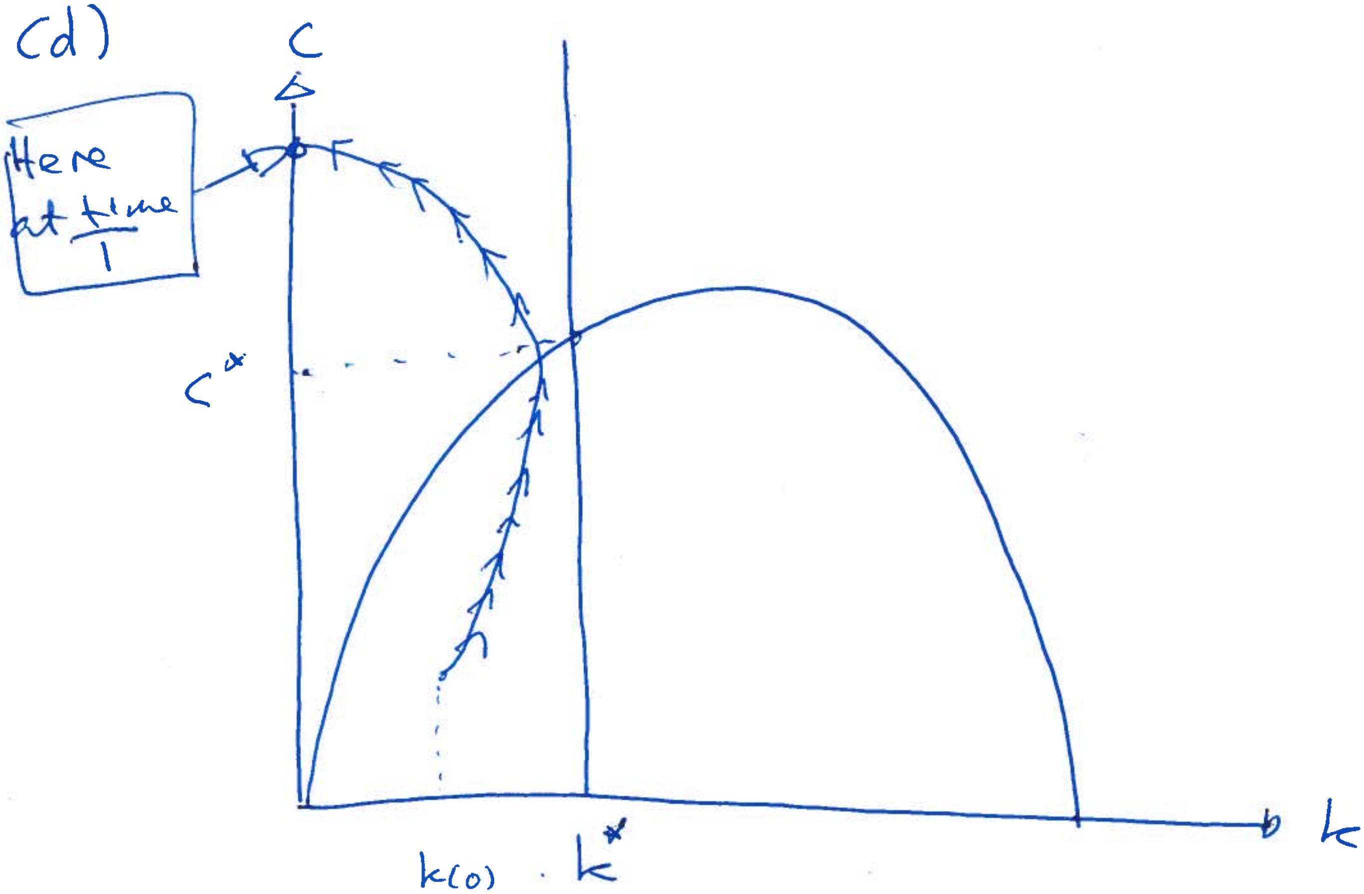
(b)



(c)



(d)



(e)