Midterm Exam – Econ 4020 v1
19 March 2018
Department of Economics
York University

Instructions: On this exam, you should *not* show how you arrived at your answer, only write the answer in the space indicated on the relevant answer sheet. Answers must be given on the correct answer sheet, one for each problem. If you make a mistake, ask for a new answer sheet. Do not fold the answer sheets or write on the back, or outside of the indicated space.

Student name: SID number:

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1. The Solow model [3 marks]

Consider a Solow model with CES production, meaning output per effective worker is given by

$$f(k) = (\alpha k^{\rho} + 1 - \alpha)^{\frac{1}{\rho}},$$

where k denotes the capital stock per effective worker, and where $0 < \alpha < 1$, and $\rho < 1$. The capital stock per effective worker evolves over time according to

$$\dot{k} = sf(k) - (n + g + \delta)k,$$

where n, g, and δ are all strictly positive, and $0 < s < 1$. Let consumption per effective worker be denoted $c = (1 - s)f(k)$. Note that c and k depend on time, t, but we here suppress the time argument.

(a) Find an expression for the steady-state level of k, denoted k^*. Your answer should be in terms of (some or all of) α, ρ, n, g, s, and δ. (We assume that the parameter values are such that a steady state exists.) [1 mark]

(b) Now assume that $0 < \rho < 1$. (Before we just assumed $\rho < 1$.) Draw the graphs of $f(k)$, $sf(k)$, and $(n + g + \delta)k$ in the diagram provided. For full mark, you also need to indicate the following in the diagram: (i) the steady-state level of k (denoted k^*); (ii) the steady-state level of c (denoted c^*); and (iii) the vertical intercepts of $f(k)$ and $sf(k)$. [2 marks]

2. The Ramsey model [4 marks]

Consider the Ramsey model. Utility of a household is given by

$$U = \int_0^{\infty} e^{-\rho t} \left(\frac{[C(t)]^{1-\theta}}{1-\theta} \right) \frac{L(t)}{H} \, dt,$$

where ρ is the discount rate, $L(t)$ is the size of the population (and workforce) at time t, $\theta > 0$ is a utility parameter, $H > 0$ is the number of households, and $C(t)$ is consumption per worker at time t. We assume that $\overset{\bullet}{L}(t) = nL(t)$, where $n > 0$ and $L(0) > 0$. Let $c(t) = C(t)/A(t)$ be consumption per effective worker, where $A(t)$ is efficiency units per worker. We assume that $\overset{\bullet}{A}(t) = gA(t)$, where $g > 0$ and $A(0) > 0$.

As seen in class, and in the textbook, we can rewrite the utility function as

$$U = B \int_0^\infty e^{-\beta t} \left(\frac{[c(t)]^{1-\theta}}{1-\theta} \right) dt.$$

(a) Find an expression for B. Your answer should be in terms of (some or all of) $L(0)$, $A(0)$, H, n, g, θ, and ρ. [1 mark]

(b) Find an expression for β. Your answer should be in terms of (some or all of) $L(0)$, $A(0)$, H, n, g, θ, and ρ. [1 mark]

(c) Assume that the interest rate is exogenous and constant over time, denoted $\overline{r} > 0$. Assume also that agents earn no labor income. The budget constraint can then be written as $k(0) = \int_0^\infty e^{(n+g-\overline{r})t} c(t)\, dt$, where $k(0)$ is the initial capital stock per effective worker. Find the Euler equation. Your answer should be an expression for $\overset{\bullet}{c}(t)/c(t)$ in terms of (some or all of) $\overline{r}$, n, g, θ, and ρ. [1 mark]

(d) Use your answer under (c), and $c(t) = C(t)/A(t)$, to find an expression for $C(t)$ in terms of (some or all of) $C(0)$, $\overline{r}$, n, g, θ, ρ, and t. [1 mark]

3. The Diamond model [5 marks]

Consider the Diamond model, where agents live for two period. They earn labor income, denoted w_t, in the first period of life, and live off savings with interest when old. (We here set the labor productivity factor A_t to 1 in all periods, i.e., $g = 0$.) The size of the young population in period t, denoted L_t, grows at rate n, i.e., $L_{t+1} = (1+n)L_t$. Capital per worker is denoted $k_t = K_t/L_t$, where K_t is the total capital stock. We let production be Cobb-Douglas and assume full depreciation, so that $w_t = (1-\alpha)k_t^\alpha$, and $1 + r_t = \alpha k_t^{\alpha-1}$, where $0 < \alpha < 1$.

First-period per-worker income is spent on saving, denoted S_t, and first-period consumption, denoted $C_{1,t}$. The first-period budget constraint can thus be written

$$C_{1,t} = w_t - S_t.$$

In retirement, the same agent consumes $C_{2,t+1}$, consisting of savings from working age, plus interest:

$$C_{2,t+1} = S_t(1 + r_{t+1}).$$

The utility function is given by

$$U_t = (1-\lambda)\ln\left(C_{1,t} - \widehat{C}\right) + \lambda \ln\left(C_{2,t+1}\right),$$

where $0 < \lambda < 1$, and $\widehat{C} > 0$. We call $\widehat{C}$ subsistence consumption.

(a) Assume that $w_t > \widehat{C}$. Derive an expression for optimal S_t in terms of (some or all of) w_t, r_{t+1}, n, $\widehat{C}$, and λ. [1 mark]

(b) If $w_t \leq \widehat{C}$, utility is technically not defined. Here we assume that $S_t = 0$ when $w_t \leq \widehat{C}$. Derive an expression for $\widehat{k}$, such that $S_t > 0$ when $k_t > \widehat{k}$, and $S_t = 0$ when $k_t \leq \widehat{k}$.

Your answer should be an expression for $\widehat{k}$ in terms of (some or all of) n, α, $\widehat{C}$, and λ. [1 mark]

(c) Given the assumptions made under (b), and by using $K_{t+1} = S_t L_t$, we can derive a function that defines k_{t+1} in terms of k_t and exogenous variables. Your task now is to draw the graph of that function in the 45-degree diagram provided. It turns out that there may be three steady states in this model. One is given by $k_{t+1} = k_t = 0$ (at the origin). The other two (strictly positive) steady-state levels of k_t are denoted $k^*$and k^{**}, where $0 < k^* < k^{**}$. For full mark you need to draw the graph so that all these three steady states exist, and indicate $\widehat{k}$, k^*, and k^{**} on suitable axis. [2 marks]

(d) Suppose an economy starts off with an initial capital stock per effective worker equal to k_0, such that $\widehat{k} < k_0 < k^*$, where the notation and assumptions are the same as in (c) above. In the diagram provided, show the time path of k_t. Indicate the levels of $\widehat{k}$, k^*, k^{**}, k_0, k_1, k_2, and k_3 on suitable axis. [1 mark]

Midterm Exam – Econ 4020 v2
19 March 2018
Department of Economics
York University

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Student name: SID number:

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1. The Solow model [3 marks]

Consider a Solow model with CES production, meaning output per effective worker is given by

$$f(k) = Zk^{\alpha},$$

where k denotes the capital stock per effective worker, and where $0 < \alpha < 1$, and $Z > 0$. The capital stock per effective worker evolves over time according to

$$\dot{k} = sf(k) - (n + g + \delta)k,$$

where n, g, and δ are all strictly positive, and $0 < s < 1$. Let consumption per effective worker be denoted $c = (1 - s)f(k)$. Note that c and k depend on time, t, but we here suppress the time argument.

(a) Find an expression for the steady-state level of c, denoted c^*. Your answer should be in terms of (some or all of) α, Z, n, g, s, and δ. [1 mark]

(b) Draw the graphs of $f(k)$, $sf(k)$, and $(n + g + \delta)k$ in the diagram provided. For full mark, you also need to indicate the following in the diagram: (i) the steady-state level of k (denoted k^*); and (ii) the steady-state level of c (denoted c^*); and (iii) the slope of $f(k)$ at k^*, i.e., $f'(k^*)$. [2 marks]

2. The Ramsey model [4 marks]

Consider the Ramsey model. Utility of a household is given by

$$U = \int_0^{\infty} e^{-\rho t} \left(\frac{[C(t)]^{1-\theta}}{1-\theta} \right) \frac{L(t)}{N} \, dt,$$

where ρ is the discount rate, $L(t)$ is the size of the population (and workforce) at time t, $\theta > 0$ is a utility parameter, $N > 0$ is the number of households, and $C(t)$ is consumption per worker at time t. We assume that $\dot{L}(t) = nL(t)$, where $n > 0$ and $L(0) > 0$. Let

$c(t) = C(t)/A(t)$ be consumption per effective worker, where $A(t)$ is efficiency units per worker. We assume that $\dot{A}(t) = gA(t)$, where $g > 0$ and $A(0) > 0$.

As seen in class, and in the textbook, we can rewrite the utility function as

$$U = B \int_0^\infty e^{-\beta t} \left(\frac{[c(t)]^{1-\theta}}{1-\theta} \right) dt.$$

(a) Find an expression for B. Your answer should be in terms of (some or all of) $L(0)$, $A(0)$, N, n, g, θ, and ρ. [1 mark]

(b) Find an expression for β. Your answer should be in terms of (some or all of) $L(0)$, $A(0)$, N, n, g, θ, and ρ. [1 mark]

(c) Assume that the interest rate is exogenous and constant over time, denoted $\overline{r} > 0$. Assume also that agents earn no labor income. The budget constraint can then be written as $k(0) = \int_0^\infty e^{(n+g-\overline{r})t} c(t)\, dt$, where $k(0)$ is the initial capital stock per effective worker. Find the Euler equation. Your answer should be an expression for $\dot{c}(t)/c(t)$ in terms of (some or all of) $\overline{r}$, n, g, θ, and ρ. [1 mark]

(d) Use your answer under (c), and $c(t) = C(t)/A(t)$, to find an expression for $C(t)$ in terms of (some or all of) $C(0)$, $\overline{r}$, n, g, θ, ρ, and t. [1 mark]

3. The Diamond model [5 marks]

Consider the Diamond model, where agents live for two period. They earn labor income, denoted w_t, in the first period of life, and live off savings with interest when old. (We here set the labor productivity factor A_t to 1 in all periods, i.e., $g = 0$.) The size of the young population in period t, denoted L_t, grows at rate n, i.e., $L_{t+1} = (1+n)L_t$. Capital per worker is denoted $k_t = K_t/L_t$, where K_t is the total capital stock. We let production be Cobb-Douglas and assume full depreciation, so that $w_t = (1-\alpha)k_t^\alpha$, and $1 + r_t = \alpha k_t^{\alpha-1}$, where $0 < \alpha < 1$.

First-period per-worker income is spent on saving, denoted S_t, and first-period consumption, denoted $C_{1,t}$. The first-period budget constraint can thus be written

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The utility function is given by

$$U_t = (1-\beta)\ln\left(C_{1,t} - \widehat{C}\right) + \beta \ln\left(C_{2,t+1}\right),$$

where $0 < \beta < 1$, and $\widehat{C} > 0$. We call $\widehat{C}$ subsistence consumption.

(a) Assume that $w_t > \widehat{C}$. Derive an expression for optimal S_t in terms of (some or all of) w_t, r_{t+1}, n, $\widehat{C}$, and β. [1 mark]

(b) If $w_t \le \widehat{C}$, utility is technically not defined. Here we assume that $S_t = 0$ when $w_t \le \widehat{C}$. Derive an expression for $\widehat{k}$, such that $S_t > 0$ when $k_t > \widehat{k}$, and $S_t = 0$ when $k_t \le \widehat{k}$. Your answer should be an expression for $\widehat{k}$ in terms of (some or all of) n, α, $\widehat{C}$, and β. [1 mark]

(c) Given the assumptions made under (b), and by using $K_{t+1} = S_t L_t$, we can derive a function that defines k_{t+1} in terms of k_t and exogenous variables. Your task now is to draw the graph of that function in the 45-degree diagram provided. It turns out that there may be three steady states in this model. One is given by $k_{t+1} = k_t = 0$ (at the origin). The other two (strictly positive) steady-state levels of k_t are denoted $k^*$and k^{**}, where $0 < k^* < k^{**}$. For full mark you need to draw the graph so that all these three steady states exist, and indicate $\widehat{k}$, k^*, and k^{**} on suitable axis. [2 marks]

(d) Suppose an economy starts off with an initial capital stock per effective worker equal to k_0, such that $k^* < k_0 < k^{**}$, where the notation and assumptions are the same as in (c) above. In the diagram provided, show the time path of k_t. Indicate the levels of $\widehat{k}$, k^*, k^{**}, k_0, k_1, k_2, and k_3 on suitable axis. [1 mark]

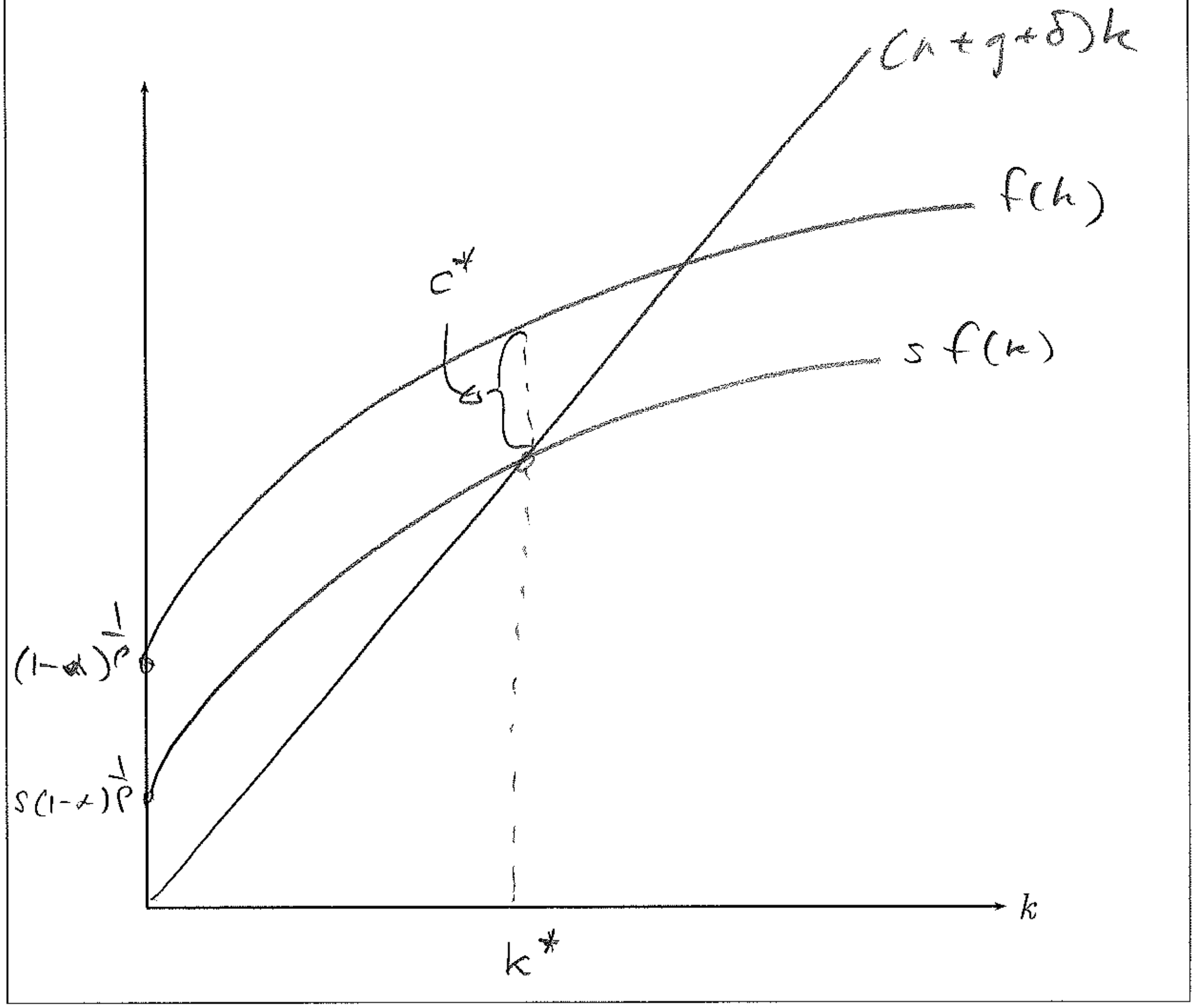
Econ 4020 Midterm Exam 19 March 2018 v1 – Answer sheet for Problem 1

Student Name:	SID Number:

Answer to 1 (a):

$$k^* = \left[\frac{(1-\alpha)s^\rho}{(n+g+\delta)^\rho - \alpha s^\rho}\right]^{\frac{1}{\rho}}$$

Answer to 1 (b):



Econ 4020 Midterm Exam 19 March 2018 v2 – Answer sheet for <u>Problem 1</u>

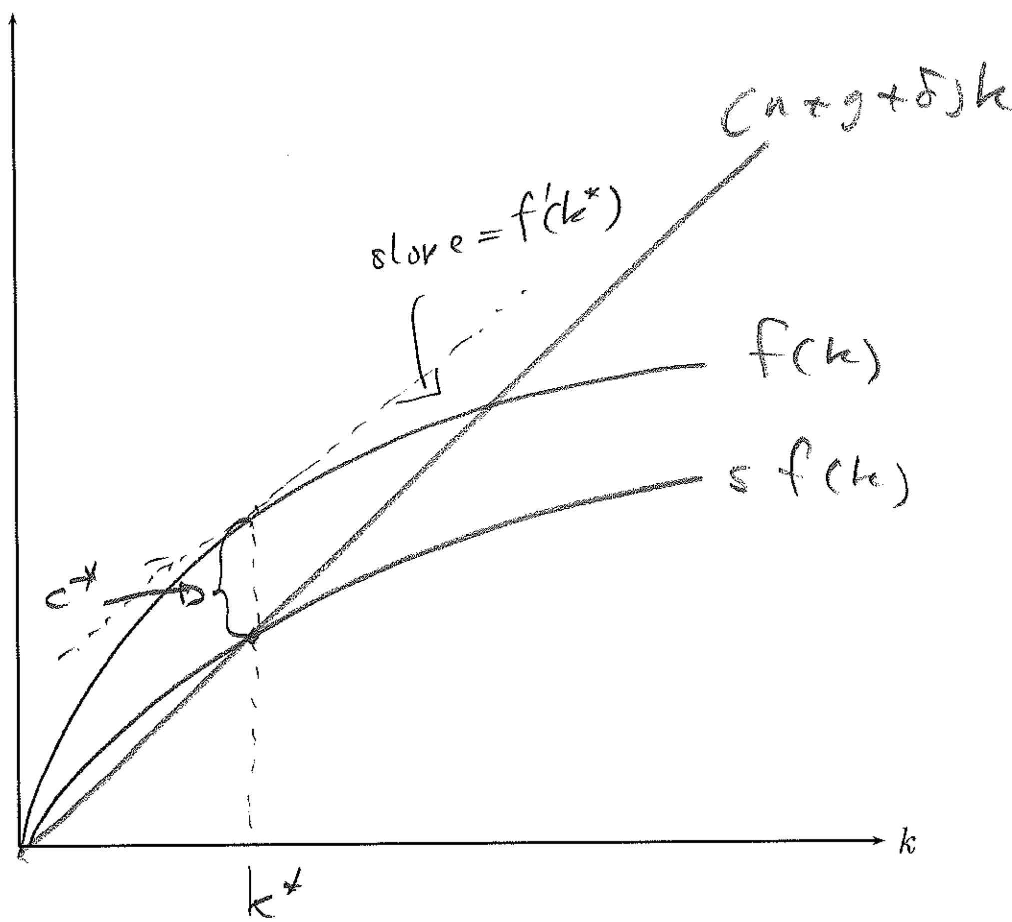
Student Name:

SID Number:

Answer to 1 (a):

$$c^* = (1-s)\, z^{\frac{1}{1-\alpha}} \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Answer to 1 (b):



Econ 4020 Midterm Exam 19 March 2018 v1 & v2 – Answer sheet for <u>Problem 2</u>

Student Name:	SID Number:

Answer to 2 (a):

$$B = \frac{[A(0)]^{1-\theta} L_1(0)}{H}$$

Answer to 2 (b):

$$\beta = \rho - n - (1-\theta)g$$

Answer to 2 (c):

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[r - \rho - \theta g\right]$$

Answer to 2 (d):

$$C(t) = C_1(0)\, e^{\left(\frac{r-\rho}{\theta}\right)t}$$

Econ 4020 Midterm Exam 19 March 2018 v1 & v2 Answer sheet for <u>Problem 2</u>

Student Name:	SID Number:

Answer to 2 (a):

$$B = \frac{[A(0)]^{1-\theta} L(0)}{N}$$

Answer to 2 (b):

$\beta =$ same as v1

Answer to 2 (c):

$\frac{\dot{c}(t)}{c(t)} =$ same as v1

Answer to 2 (d):

$C(t) =$ same as v1

Econ 4020 Midterm Exam 19 March 2018 v1 & v2 – Answer sheet for <u>Problem 3</u>

Student Name:	SID Number:

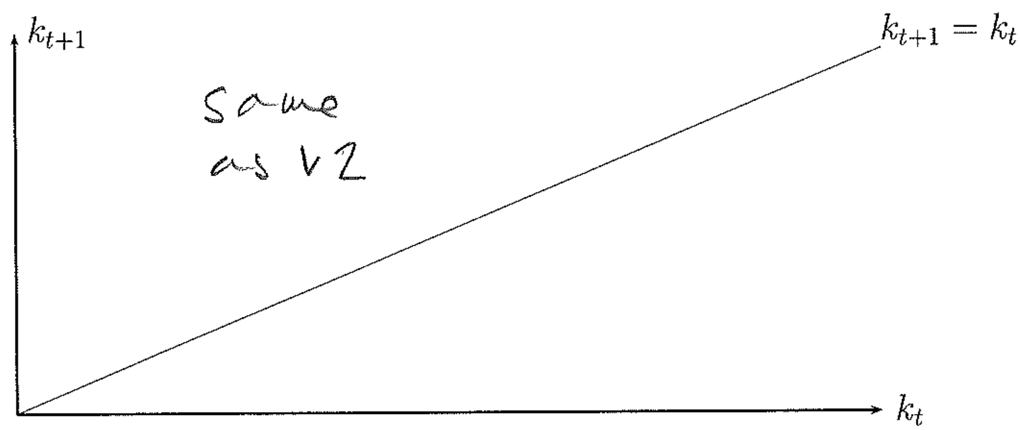
Answer to 3 (a):

$$S_t = \lambda\left[w_t - \hat{d}_1\right]$$

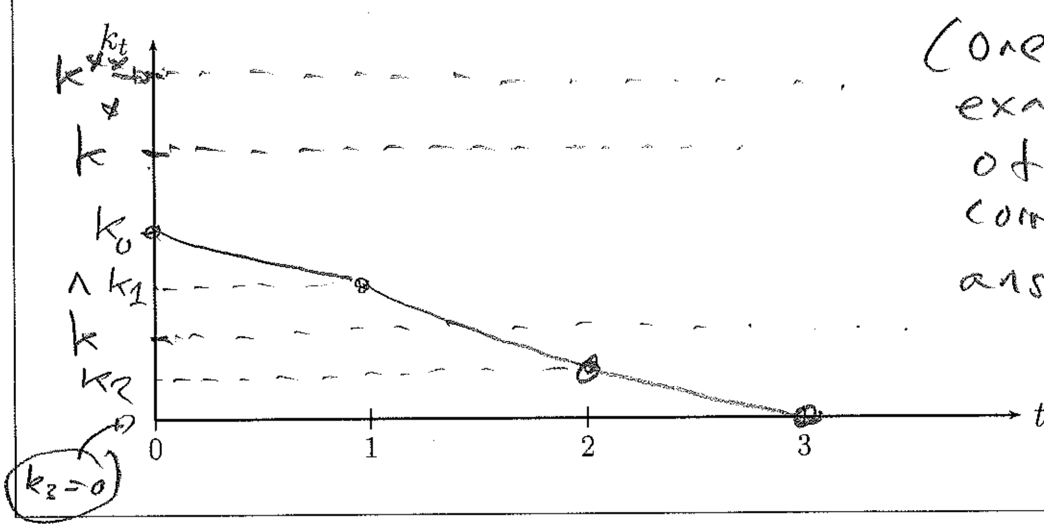
Answer to 3 (b):

$\hat{k} =$ same as v2

Answer to 3 (c):



Answer to 3 (d):



(One example of correct answer)

Econ 4020 Midterm Exam 19 March 2018 v1 & v2 Answer sheet for Problem 3

Student Name: SID Number:

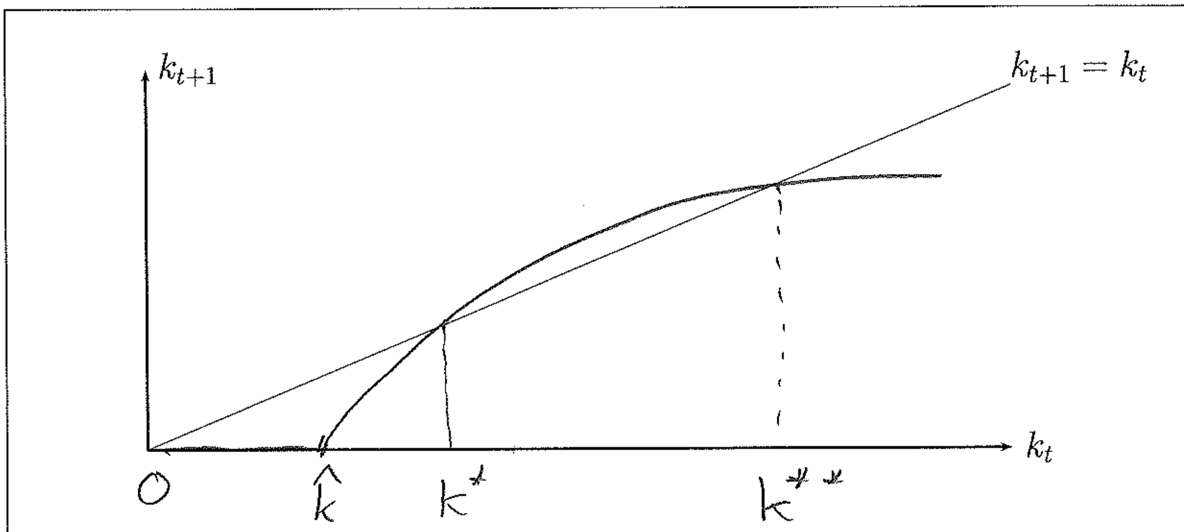
Answer to 3 (a):

$$S_t = \beta\left[w_t - \hat{c}_1\right]$$

Answer to 3 (b):

$$\hat{k} = \left(\frac{\hat{c}_1}{1-\alpha}\right)^{\frac{1}{\alpha}}$$

Answer to 3 (c):



Answer to 3 (d):