Midterm Exam – Econ 4020 – v1
3 February 2020
Department of Economics
York University

Instructions: On this exam, you should *not* show how you arrived at your answer, only write the answer in the space indicated on the relevant answer sheet. Answers must be given on the correct answer sheet, one for each problem. If you make a mistake, ask for a new answer sheet. Do not fold the answer sheets or write on the back, or outside of the indicated space.

Student name: SID number:

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1. Capital stock dynamics and specific production functions [**4 marks**]

Consider the Solow model with depreciation (δ) set to zero. The total capital stock (K) then evolves according to

$$\overset{\bullet}{K} = sY,$$

where Y denotes total output, and $s \in (0,1)$ is the rate of saving (and investment). Output is given by the Neoclassical production function

$$Y = F(K, AL),$$

where A and L denote labor-augmenting productivity, and the size of the labor force, which grow at rates g and n, respectively. Note that s, n, and g, are all strictly positive, exogenous, and constant over time, while Y, K, A and L are functions of time, although we suppress the time argument. Lower-case letters denote units per effective worker, so that $k = K/(AL)$, and $y = Y/(AL)$.

The real interest rate, denoted r, equals the marginal product of capital:

$$r = \frac{\partial F(K, AL)}{\partial K}.$$

For (a) and (b) below, let $Y = K^{\alpha}(AL)^{1-\alpha}$, where $\alpha \in (0,1)$.

(a) Find an expression for $\overset{\bullet}{k}$ in terms of (some or all of) k, s, n, α, and g (and possibly numbers). [1 mark]

(b) Find an expression for $\overset{\bullet}{r}$ in terms of (some or all of) r, s, n, α, and g (and possibly numbers). [1 mark]

For (c) and (d) below, $Y = [(1-\beta)K^{\rho} + \beta(AL)^{\rho}]^{\frac{1}{\rho}}$, where $\beta \in (0,1)$, $\rho < 1$, and $\rho \neq 0$.

(c) Find an expression for the steady-state level of y, denoted y^* (assuming it exists). Your answer should be in terms of (some or all of) s, n, β, ρ, and g (and possibly numbers). [1 mark]

(d) Find an expression for the steady-state level of r, denoted r^* (assuming it exists). Your answer should be in terms of (some or all of) s, n, β, ρ, and g (and possibly numbers). [1 mark]

2. Growth accounting [4 marks]

Consider this (extensive-form) production function:

$$Y = F(BL, K),$$

where Y is total output, K is the total capital stock, L is the total labor force, and B is a labor-augmenting productivity factor, all dependent on time (t), but we suppress the time argument.

Note that the order of the two arguments in F differs from what we did in class.

Let $k = K/(BL)$ and $y = Y/(BL)$. We assume that F exhibits Constant Returns to Scale, implying that there exists an intensive-form production function, f, such that $y = f(k)$. Also, let $\alpha(k) = f'(k)k/f(k)$.

(a) Define f in terms of F. Your answer should be an expression for $f(k)$ involving F and k (and numbers). [1 mark]

(b) Find an expression for Y involving f, K, B, and L (but not F). [1 mark]

(c) Find an expression for $\dot{Y}/Y$ in terms of $\alpha(k)$, $\dot{L}/L$, $\dot{K}/K$ and $\dot{B}/B$. [1 mark]

(d) Find an expression for $\dot{y}/y$ in terms of $\alpha(k)$, $\dot{L}/L$, $\dot{K}/K$ and $\dot{B}/B$. [1 mark]

3. Time paths and the golden rule [4 marks]

Consider the Solow model again, where the total capital stock is denoted K, and the capital stock per effective worker equals $k = K/(AL)$, where A and L denote labor-augmenting productivity and the size of the labor force, and grow at rates g and n, respectively. As usual, k evolves according to

$$\dot{k} = sf(k) - (n + g + \delta)k,$$

where n, g, and δ are all strictly positive, and where $s \in (0, 1)$, $f'(k) > 0$, and $f''(k) < 0$. Consumption per effective worker equals $c = (1 - s)f(k)$. Note that c, k, K, A, and L all depend on time, t, but we suppress the time argument.

Assume that the economy is initially in a steady state associated with $s = s_0$. Then s changes from s_0 to s_1, from some point in time $\widehat{t}$, and then stays at s_1 forever. We also assume that

$$s_0 < s_1 < s_{\mathrm{GR}},$$

where s_{GR} is the golden-rule level of s, which maximizes steady-state consumption per effective worker, c^*. Let k_0^* and k_1^* be the steady-state levels of k associated with s_0 and s_1, respectively. Similarly, let c_0^* and c_1^* be the steady-state levels of c associated with s_0 and s_1, respectively, and let c_{GR}^* be the steady-state level of c associated with s_{GR}.

(a) Letting the time argument be explicit we can write growth in labor-augmenting productivity as $\dot{A}(t)/A(t) = g$. Find an expression for $A(t)$ in terms of (some or all of) $A(0)$, t, g, n, δ, s_0, s_1, and the number e (the base of the exponential function). [1 mark]

(b) In the diagram provided, draw the time path for c. For full mark you must: (i) draw the path correctly; and (ii) indicate c_0^*, c_1^*, c_{GR}^*, and $\widehat{t}$ correctly on the relevant axes. [1 marks]

(c) In the diagram provided, draw the time path for the log of the total capital stock, $\ln(K)$. For full mark you must: (i) draw the path correctly; (ii) indicate $\widehat{t}$ on the relevant axis; and (iii) indicate the slope of the path, both before $\widehat{t}$ and as t goes to infinity. [2 marks]

Midterm Exam – Econ 4020 – v2
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The real interest rate, denoted r, equals the marginal product of capital:

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For (a) and (b) below, let $Y = K^{1-\beta}(AL)^{\beta}$, where $\beta \in (0,1)$.

(a) Find an expression for $\dot{k}$ in terms of (some or all of) k, s, n, β, and g (and possibly numbers). [1 mark]

(b) Find an expression for $\dot{r}$ in terms of (some or all of) r, s, n, β, and g (and possibly numbers). [1 mark]

For (c) and (d) below, $Y = [\alpha K^{\rho} + (1-\alpha)(AL)^{\rho}]^{\frac{1}{\rho}}$, where $\alpha \in (0,1)$, $\rho < 1$, and $\rho \neq 0$.

(c) Find an expression for the steady-state level of y, denoted y^* (assuming it exists). Your answer should be in terms of (some or all of) s, n, α, ρ, and g (and possibly numbers). [1 mark]

(d) Find an expression for the steady-state level of r, denoted r^* (assuming it exists). Your answer should be in terms of (some or all of) s, n, α, ρ, and g (and possibly numbers). [1 mark]

2. Growth accounting [4 marks]

Consider this (extensive-form) production function:

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(b) Find an expression for Y involving f, K, A, and L (but not F). [1 mark]

(c) Find an expression for $\dot{Y}/Y$ in terms of $\alpha(k)$, $\dot{L}/L$, $\dot{K}/K$ and $\dot{A}/A$. [1 mark]

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Assume that the economy is initially in a steady state associated with $s = s_0$. Then s changes from s_0 to s_1, from some point in time $\widehat{t}$, and then stays at s_1 forever. We also assume that

$$s_{\text{GR}} < s_0 < s_1,$$

where s_{GR} is the golden-rule level of s, which maximizes steady-state consumption per effective worker, c^*. Let k_0^* and k_1^* be the steady-state levels of k associated with s_0 and s_1, respectively. Similarly, let c_0^* and c_1^* be the steady-state levels of c associated with s_0 and s_1, respectively, and let c_{GR}^* be the steady-state level of c associated with s_{GR}.

(a) Letting the time argument be explicit we can write growth in labor-augmenting productivity as $\dot{A}(t)/A(t) = g$. Find an expression for $A(t)$ in terms of (some or all of) $A(0)$, t, g, n, δ, s_0, s_1, and the number e (the base of the exponential function). [1 mark]

(b) In the diagram provided, draw the time path for c. For full mark you must: (i) draw the path correctly; and (ii) indicate c_0^*, c_1^*, c_{GR}^*, and $\widehat{t}$ correctly on the relevant axes. [1 marks]

(c) In the diagram provided, draw the time path for the log of the total capital stock, $\ln(K)$. For full mark you must: (i) draw the path correctly; (ii) indicate $\widehat{t}$ on the relevant axis; and (iii) indicate the slope of the path, both before $\widehat{t}$ and as t goes to infinity. [2 marks]

Sketches of solutions - v1

Problem 1:
(a)

$$\dot{k} = sk^{\alpha} - (n+g)k$$

(b)

$$\dot{r} = (1-\alpha)\left[n + g - \frac{sr}{\alpha}\right] r$$

To solve (b), use $r = \alpha k^{\alpha-1}$, which implies $k^{\alpha}/k = r/\alpha$ and $\dot{r}/r = (\alpha - 1)\dot{k}/k$; then use solution to (a).
(c)

$$y^* = (n+g)\left[\frac{\beta}{(n+g)^{\rho} - (1-\beta)s^{\rho}}\right]^{\frac{1}{\rho}}$$

(d)

$$r^* = (1-\beta)\left(\frac{n+g}{s}\right)^{1-\rho}$$

To solve (d) you may show that $r = (1-\beta)(k/y)^{\rho-1}$, and use $k^*/y^* = s/(n+g)$.

Problem 2:
(a)

$$f(k) = F(1, k)$$

(b)

$$Y = BLf\left(\frac{K}{BL}\right)$$

(c)

$$\frac{\dot{Y}}{Y} = [1 - \alpha(k)]\left(\frac{\dot{B}}{B} + \frac{\dot{L}}{L}\right) + \alpha(k)\frac{\dot{K}}{K}$$

(d)

$$\frac{\dot{y}}{y} = \alpha(k)\left(\frac{\dot{K}}{K} - \frac{\dot{B}}{B} - \frac{\dot{L}}{L}\right)$$

Problem 3:
(a)

$$A(t) = A(0)e^{gt}$$

(b) First c equals c_0^* until $\widehat{t}$. Then there is a fall in c at $\widehat{t}$, then a gradual rise in c, as c converges to the new and higher steady-state level, $c_1^* > c_0^*$. It holds that $c_0^* < c_1^* < c_{\text{GR}}^*$, since $s_0 < s_1 < s_{\text{GR}}$.

(c) First the graph of $\ln(K)$ is a straight line with slope $n + g$ until $\widehat{t}$. Then the slope becomes temporarily steeper (but with finite slope) right after $\widehat{t}$. From then the slope of the graph of $\ln(K)$ declines and asymptotically approaches the original slope, $n + g$.

Sketches of solutions - v2

Problem 1:
(a)-(b) Same as v1 but α replaced by $1-\beta$.
(c)-(d) Same as v1 but β replaced by $1-\alpha$.

Problem 2: Same as v1 but B replaced by A.

Problem 3:
(a) and (c): Same as v1.
(b) First c equals c_0^* until $\widehat{t}$. Then there is a fall in c at $\widehat{t}$, then a gradual rise in c, as c converges to the new steady-state level, c_1^*, which is below the original one, i.e., $c_1^* < c_0^*$. It holds that $c_{\mathrm{GR}}^* > c_0^* > c_1^*$, since $s_{\mathrm{GR}} < s_0 < s_1$.