Some lecture notes for Econ 4020

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1 Introduction

1.1 Production functions

The usual way to model the production process in macroeconomics is by using a *production function*. This tells us the amount that can be produced of some good, called *output*, as a function of the amounts of various types of production factors used in the production process, called *inputs*. There may be several types of inputs, but usually only one output. For example, output could be the amount of potato that can be produced given several different types of input, such as labor, land, and capital.

Many textbook models describe an economy with only one good, that is produced using only two inputs, capital and labor. (Problem 1 below considers an example with three production inputs.) Let Y denote output and let K and L denote the inputs of capital and labor, respectively. Y, K, and L are all variables, and represent numbers. Because we cannot think of negative levels of output, capital, or labor, it is implicitly assumed that $Y \geq 0$, $K \geq 0$, and $L \geq 0$.

The production function is denoted by F. This is a function, not a variable. It determines output for given levels of the two inputs, as follows:

$$Y = F(K, L). \tag{1}$$

If the single good is the potato, then Y denotes the amount of potato harvested, K is the amount of potato planted in the soil, and L could be the number of workers employed in potato harvesting, or the number of hours worked.

In some formulations we write $Y = F(K, AL)$, where AL denotes "efficient labor." Note that the function still has only two arguments, K and AL. The second argument, AL, is the product of two variables: L is the number of workers (or hours worked), while A is a productivity factor, which we would call labor-augmenting productivity (i.e., it "augments" labor).

Sometimes you may see the production function in (1) defined as $F\colon \mathbb{R}_+^2 \to \mathbb{R}_+$. In this example, $\mathbb{R}_+^2$ is the *domain* of F, i.e., the set that F maps from, while $\mathbb{R}_+$ is the *range* of F, i.e., the set it maps to. $\mathbb{R}_+$ is the non-negative real line. This is the set of numerical values that Y can take, i.e., all numbers from 0 and up. $\mathbb{R}_+^2$ is the two-dimensional space on which the two inputs, K and L, can fall, i.e., all combinations of any two non-negative real numbers. In words, one might say something like: "F is a function that maps from the two-dimensional space of non-negative real numbers to a point on the non-negative real line."

In this example, the domain is often illustrated by a standard two-dimensional diagram with the two inputs K and L on the axes. A line (or curve) that connects combinations of

K and L that produce a given level of Y is called an *isoquant*.

1.1.1 Common assumptions

Throughout this course, we will almost always make the following assumptions about F.

1. F exhibits positive marginal products. This means that F is increasing in both its arguments:

$$\begin{aligned} \tfrac{\partial F(K,L)}{\partial K} &> 0, \\ \tfrac{\partial F(K,L)}{\partial L} &> 0. \end{aligned} \tag{2}$$

2. F exhibits diminishing marginal products. This means that the first derivative of F with respect to each of the two arguments is decreasing in that respective argument:

$$\begin{aligned} \tfrac{\partial^2 F(K,L)}{\partial K^2} &< 0, \\ \tfrac{\partial^2 F(K,L)}{\partial L^2} &< 0. \end{aligned} \tag{3}$$

3. F exhibits Constant Returns to Scale (CRS). This means that scaling each input by some strictly positive factor leads to a change in output by the same factor. That is, for any $\lambda > 0$ it holds that

$$\lambda F(K, L) = F(\lambda K, \lambda L). \tag{4}$$

   For example, a doubling of both inputs ($\lambda = 2$) leads to a doubling of output.

There are two more assumption that we usually make, but not always.

4. All inputs in F are necessary for strictly positive output. That is,

$$\begin{aligned} F(K, 0) = 0 \quad &\text{for all } K \geq 0, \\ F(0, L) = 0 \quad &\text{for all } L \geq 0. \end{aligned} \tag{5}$$

5. F satisfies the Inada conditions. That is,

$$\begin{aligned} \lim_{K \to +\infty} \tfrac{\partial F(K,L)}{\partial K} = 0 \quad &\text{for all } L > 0, \\ \lim_{L \to +\infty} \tfrac{\partial F(K,L)}{\partial L} = 0 \quad &\text{for all } K > 0, \\ \lim_{K \to 0} \tfrac{\partial F(K,L)}{\partial K} = +\infty \quad &\text{for all } L > 0, \\ \lim_{L \to 0} \tfrac{\partial F(K,L)}{\partial L} = +\infty \quad &\text{for all } K > 0. \end{aligned} \tag{6}$$

Production functions that satisfy Assumptions 1-5 are often called *Neoclassical* production functions. You may encounter texts which define a Neoclassical production function as one that satisfies Assumptions 1-3, but not necessarily 4 and 5. For the rest of this course, I will probably use the term Neoclassical production function to imply one where all five assumptions hold.

1.1.2 The intensive-form production function

Suppose a production function F exhibits CRS, as defined in Assumption 3 above. Let k and y denote capital and output per worker, i.e., $k = K/L$, and $y = Y/L$. Then there exists another production function, f, which we call the intensive-form production function, such that

$$y = f(k). \tag{7}$$

To see this, we can set $\lambda = 1/L$ in the definition of CRS above, which gives $Y/L = F(K/L, 1)$, or $y = F(k, 1)$. The function f is thus defined from

$$f(k) = F(k, 1). \tag{8}$$

Note that (8) defines f in terms of F. For example, $f(5)$ by definition gives the same value as $F(5, 1)$. Note also that f has one argument while F has two. We are not just dropping one argument of F, but defining a different function. It is thus wrong to write, e.g., "$F(k)$" or "$f(K, L)$" in this context.

Some properties of f follow from Assumptions 1 and 2 above. In particular,

$$\begin{aligned} f'(k) &= \tfrac{\partial F(k,1)}{\partial k} > 0, \\ f''(k) &= \tfrac{\partial^2 F(k,1)}{\partial k^2} < 0. \end{aligned} \tag{9}$$

Moreover, Assumption 4 implies

$$f(0) = F(0, 1) = 0. \tag{10}$$

Finally, Assumption 5 implies

$$\begin{aligned} \lim\nolimits_{k \to +\infty} f'(k) &= 0, \\ \lim\nolimits_{k \to 0} f'(k) &= +\infty. \end{aligned} \tag{11}$$

We can now draw the graph of $y = f(k)$ in a diagram with k on the horizontal axis and y on the vertical axis. The graph should have positive and diminishing slope, start at the origin with infinite slope, and have zero slope asymptotically.

We can also draw the graph of $f'(k)$ in a diagram with k on the horizontal axis. Note that graph should have negative slope, since $f''(k) < 0$. Since $f(0) = 0$, we can also write

$$f(k) = \int_0^k f'(x)dx, \tag{12}$$

which can be illustrated as the surface under the graph of $f'(x)$ from $x = 0$ to $x = k$. (Note that we must label the variable that we integrate over, x, something distinct from the upper limit of the integral, k.) It can be shown that $f''(k) < 0$ implies $f'(k)k < f(k)$ (use the diagram to illustrate). Rearranging, we see that

$$\frac{f'(k)k}{f(k)} < 1. \tag{13}$$

We sometimes define the left-hand side of (13) as the function $\alpha(k)$, which we call the capital share of output; we return to this later.

Problem 1 *Suppose a production function F has three inputs: X, K, and L. As before, K and L denote inputs of capital and labor, respectively, and X here denotes land input. Suppose F exhibits CRS with respect to all its three inputs. Write an expression that defines what CRS means in this case, analogous to the expression in (4), which applied to a production function with only two inputs. Find an intensive-form production function, f, that defines output per worker as a function of land per worker and capital per worker. Make sure to define f in terms of F.*

1.1.3 Parametric examples

We often use production functions that are completely characterized by one or more parameters. The parameters are constant when we change production inputs or other variables of the model.

The most common example is the *Cobb-Douglas* production function. In the case with two inputs (capital and labor) this is usually written

$$Y = K^{\alpha}L^{1-\alpha}, \tag{14}$$

where $\alpha \in (0, 1)$ is a parameter, often called the capital share of output. Sometimes, we also include a productivity parameter, which we may here call $Z > 0$, such that $Y = ZK^{\alpha}L^{1-\alpha}$. In this case, Z would be called Total Factor Productivity (TFP).

The Cobb-Douglas production function is a special case where the capital share, as defined more generally by the left-hand side of (13), is constant. To see this, you can solve the following problem.

Problem 2 *Show that the production function in (14) exhibits CRS. Find the intensive-form production function, f, and show that $f'(k)k/f(k) = \alpha$.*

Another production function is the *Constant Elasticity of Substitution* (CES) production function. This can be written

$$Y = \left[\alpha K^{\rho} + (1-\alpha)L^{\rho}\right]^{\frac{1}{\rho}}, \tag{15}$$

where α and ρ are two parameters characterizing the function, which satisfy $\alpha \in (0,1)$, $\rho < 1$, and $\rho \neq 0$. That is, ρ can take any value on $(0,1)$ or $(-\infty, 0)$. As in the Cobb-Douglas case, we could also include a TFP parameter, Z, such that $Y = Z\left[\alpha K^\rho + (1-\alpha)L^\rho\right]^{\frac{1}{\rho}}$.

It is easy to verify that Assumptions 1-3 hold for a CES production function (i.e., positive and diminishing marginal products, and CRS). However, Assumptions 4 and 5 need not hold for a CES production function.

Problem 3 *Show that Assumptions 4 and 5 need not hold for the CES production function in (15).*

The parameter ρ determines the elasticity of substitution between capital and labor inputs, such that a high ρ implies higher elasticity. More precisely, the elasticity of substitution equals $1/(1-\rho) \in (0, +\infty)$.

The CES production function approaches a Cobb-Douglas production function as $\rho \to 0$ [implying $1/(1-\rho) \to 1$]. That is,

$$\lim_{\rho \to 0} \left[\alpha K^\rho + (1-\alpha)L^\rho\right]^{\frac{1}{\rho}} = K^\alpha L^{1-\alpha}. \tag{16}$$

This can be shown by taking the natural logarithm and applying l'Hôpital's rule. In other words, Cobb-Douglas is a special case of CES, namely one where the elasticity of substitution is 1.

It can also be shown that the CES production function approaches a so-called *Leontief* production function as $\rho \to -\infty$, i.e.,

$$\lim_{\rho \to -\infty} \left[\alpha K^\rho + (1-\alpha)L^\rho\right]^{\frac{1}{\rho}} = \min\{K, L\}. \tag{17}$$

In other words, with Leontief production, output equals whichever input is the smallest. If $K > L$, then $Y = \min\{K, L\} = L$, and if $K < L$, then $Y = \min\{K, L\} = K$. This implies an elasticity of substitution of 0. Intuitively, the inputs cannot be substituted for one another at all.

The problem below shows that (17) holds when $K > L$. (The case when $K < L$ is analogous, and the case when $K = L$ is trivial.)

Problem 4 *Suppose the production function is given by (15) with $\rho < 0$, and consider the case when $K > L$. Show that*

$$L < Y < (1-\alpha)^{\frac{1}{\rho}} L. \tag{18}$$

What does $(1-\alpha)^{1/\rho}$ approach as $\rho \to -\infty$? What does Y approach as $\rho \to -\infty$?

1.2 Elasticities

In economics we sometimes refer to the elasticity of one variable with respect to another. For example, let y depend on x through some function f, such that

$$y = f(x). \tag{19}$$

Then the elasticity of y with respect to x, here denoted $\varepsilon_{y,x}$ can be written as either one of these expressions:

$$\varepsilon_{y,x} = \frac{\partial y}{\partial x}\frac{x}{y} = \frac{\partial f(x)}{\partial x}\frac{x}{f(x)} = \frac{f'(x)x}{f(x)}. \tag{20}$$

The interpretation of $\varepsilon_{y,x}$ is that a 1% increase in x generates an increase in y by $\varepsilon_{y,x}$%.

It can also be seen that

$$\varepsilon_{y,x} = \frac{\partial \ln y}{\partial \ln x} = \frac{\partial \ln f(x)}{\partial \ln x}, \tag{21}$$

which is useful to know.

Problem 5 *Derive (21).*

Elasticities are used in many contexts. One reason is that we sometimes do not have a clear interpretation of either x or y, or we may not have a good comparison of x and/or y between different countries. Suppose, for example, that we model an agricultural economy, where x is rainfall, measured in millimeters per day, and y is output, measured as tons of wheat harvested. If $\varepsilon_{y,x} = 1.5$, then we can say that a 1% increase in rainfall raises the harvest by 1.5%, or that a 100% increase in rainfall increases the harvest by 150%. This makes more sense than saying that an increase in rainfall by 2 millimeters per day raises output by, say, 11 tons.

Another reason that we often use elasticities is that many relationships that we observe in the data seem to be *isoelastic*, meaning they have the same elasticity regardless of the level of input (and output). A function that exhibits an isoelastic relationship between x and y takes the form

$$y = Ax^\alpha, \tag{22}$$

for some constants A and α. It is easy to see that the Cobb-Douglas production function is isoelastic, but the CES production is not.

Moreover, if we want to illustrate the relationship between two variables with an isoelastic relationship, like that in (22), then we can use a diagram with $\ln x$ and $\ln y$ on the axes. The graph is a straight line, since $\ln y = \ln(A) + \alpha \ln x$. With the same logic, if we we want to know if a relationship in the data is isoelastic or not, then we can plot $\ln y$ and $\ln x$ in a diagram and see if the relationship looks linear.

Problem 6 *Let $y = Bz^\beta$ and $z = Cx^\gamma$. Find the elasticities of z and y with respect to x. That is, find $\varepsilon_{z,x}$ and $\varepsilon_{y,x}$.*

1.3 Continuous-time variables

Consider some variable that depends on time, $X(t)$. We let time, t, be continuous, which means that the variable t can take any non-negative real value.

We usually write the derivative of $X(t)$ with respect to t as $\dot{X}(t)$. We can also write it in a couple of other ways:

$$\dot{X}(t) = X'(t) = \frac{\partial X(t)}{\partial t}. \tag{23}$$

The latter two expressions in (23) may seem more familiar, but for some reason time derivatives are usually denoted with a dot, rather than a prime. The definition of $\dot{X}(t)$ is the same as other derivatives, i.e.,

$$\dot{X}(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}. \tag{24}$$

That $X(t)$ depends on time means that it evolves over time. Starting at some point in time t_0, it can take another value at some later point in time, $t_1 > t_0$. If $\dot{X}(t_0) > 0$, then $X(t)$ is increasing at that moment (i.e., at time $t = t_0$). If t_1 is close to t_0, then it also follows that $X(t_1) > X(t_0)$, since $X(t)$ must have increased between t_0 and t_1.

Problem 7 *Draw a graph of some $X(t)$ in a diagram with t on the horizontal axis and $X(t)$ on the vertical axis. Consider two points in time, t_0 and $t_1 > t_0$. Illustrate t_1, t_0, $X(t_1)$, $X(t_0)$, and $\dot{X}(t_0)$ in the diagram.*

We can also use *definite integrals* to express a more precise a relationship between $X(t_1)$ and $X(t_0)$. Specifically,

$$X(t_1) - X(t_0) = \int_{t_0}^{t_1} \dot{X}(t) dt. \tag{25}$$

In words, the change in $X(t)$ between t_0 and t_1 is the integral of all the incremental changes over this time interval.

A useful concept in economics is that of *growth rates*. The growth rate of $X(t)$ is simply $\dot{X}(t)/X(t)$. We can also refer to growth rates at specific points in time. For example, the growth rate of $X(t)$ at time t_0 is $\dot{X}(t_0)/X(t_0)$.

There is a special relationship between exponential and logarithmic functions on the one hand, and growth rate on the other. Using the chain rule, it can easily be seen that

$$\frac{\partial \ln X(t)}{\partial t} = \frac{\partial \ln X(t)}{\partial X(t)} \frac{\partial X(t)}{\partial t} = \left[\frac{1}{X(t)}\right] \dot{X}(t) = \frac{\dot{X}(t)}{X(t)}. \tag{26}$$

This is a useful rule, or formula, to remember. It means that we can interpret the growth rate of $X(t)$ graphically as the slope of $\ln X(t)$ in a diagram with $\ln X(t)$ on the vertical axis and t on the horizontal axis.

One class of time-dependent variable consists of those that exhibit a constant growth rate. If a variable $X(t)$ has a constant growth rate γ, $\dot{X}(t)/X(t) = \gamma$, then it takes the functional form

$$X(t) = X(0)e^{\gamma t}. \tag{27}$$

To verify this, we can first evaluate (27) at $t = 0$. This gives $X(0) = X(0)e^0 = X(0)$, which is true. Then we must also confirm that the growth rate of $X(t)$, when derived from (27), really equals γ. One quick way to see this is to first log both sides of (27) to get

$$\ln X(t) = \ln X(0) + \gamma t, \tag{28}$$

and then differentiate (28) with respect to t, recalling (26), which gives

$$\frac{\partial \ln X(t)}{\partial t} = \frac{\dot{X}(t)}{X(t)} = \frac{\partial \ln X(0)}{\partial t} + \gamma = \gamma. \tag{29}$$

Here we have used the fact that $X(0)$ is constant, meaning its time derivative is zero.

This approach started with (27) to check that it implies $\dot{X}(t)/X(t) = \gamma$. But how do we know what to guess (or conjecture) in the first place? And how do we know if this conjecture is unique?

We can instead start with the condition for a constant growth rate, $\dot{X}(t)/X(t) = \gamma$. This is a so-called *differential equation*. We are looking for an equation for $X(t)$ that solves that differential equation. To find this, we can first define a new variable, $Z(t) = \ln X(t)$. Then it follows that

$$\frac{\dot{X}(t)}{X(t)} = \frac{\partial \ln X(t)}{\partial t} = \frac{\partial Z(t)}{\partial t} = \dot{Z}(t) = \gamma. \tag{30}$$

Now we can apply (25) to see that

$$Z(t) - Z(0) = \int_0^t \dot{Z}(\tau)d\tau = \int_0^t \gamma d\tau = \gamma \times t - \gamma \times 0 = \gamma t. \tag{31}$$

(Note that we use τ to denote the variable over which we integrate. It must be called something different than the upper limit of the integral, which is here t.) Finally, we note that

$$Z(t) - Z(0) = \ln X(t) - \ln X(0) = \ln \left[\frac{X(t)}{X(0)}\right]. \tag{32}$$

Using (31) and (32) we see that

$$\exp\left[Z(t) - Z(0)\right] = \frac{X(t)}{X(0)} = \exp\left(\gamma t\right) = e^{\gamma t}. \tag{33}$$

Multiplying both sides of (33) by $X(0)$ now gives (27).

To sum up, we started with the differential equation $\dot{X}(t)/X(t) = \gamma$, and found the solution $X(t) = X(0)e^{\gamma t}$. These steps are useful to remember.

We can generalize this to the case when the growth rate of $X(t)$ is time dependent. Denote that growth rate by $r(t)$, so that $\dot{X}(t)/X(t) = r(t)$. Then we can define a new variable, $R(t)$, as follows:

$$R(t) = \int_0^t r(\tau)d\tau. \tag{34}$$

Following the same steps as above, we can show that the differential equation $\dot{X}(t)/X(t) = r(t)$ has solution

$$X(t) = X(0)e^{R(t)}. \tag{35}$$

This incorporates the special case when $r(t) = \gamma$, which implies $R(t) = \gamma t$. This is also useful to remember.

Problem 8 *Suppose $P(t)$ is the value (or price) of an asset at time t, and that the return to holding this asset at any time t must equal the (risk-free) interest rate at that point in time, denoted $r(t)$. This is what we often call a no-arbitrage condition. Suppose the asset will pay a dividend of Z at some point in time $T > 0$, and no dividend before or after that. For example, Z could be the selling price of a house that will be finished and sold at time T, and $P(t)$ the value of the building project before it is finished. Show that the asset value at time 0 equals*

$$P(0) = Ze^{-R(T)}, \tag{36}$$

where $R(T) = \int_0^T r(\tau)d\tau$. We call $P(0)$ the present value of Z.

1.4 Hazard rates and Poisson processes: an asset pricing example

Some variables in continuous-time models are stochastic, or (with a different word) random. For example, income or productivity could change over time in stochastic ways, due to shocks that we usually treat as exogenous in economic models, like wars, epidemics, or the weather.

There are many ways to model such dynamic stochastic processes. One way is to assume that a variable can take two different values and that it jumps randomly between these two values. Here we are going to consider the example of an asset which pays a high dividend in good times and a low dividend in bad times. We refer to good and bad times as two distinct *states of the world*. Two events can happen in this type of two-state model: we can go from good times to bad, and from bad times to good. We assume that these events occur at rates that are constant over time, meaning they follow a so-called *Poisson process*.

Let $b > 0$ be the constant rate at which the economy transitions from good times to bad, and let the corresponding rate at which the economy transitions from bad times to good be $g > 0$. Both b and g are referred to as a *hazard rates*. They are not quite the same as probabilities, but we can call them *instantaneous* probabilities, or probabilities *per unit of time*. (See Section B of the appendix for a more detailed explanation.)

Let $V_G(t)$ be the value (price) of the asset if the economy is in good times at time t, and $V_B(t)$ the value of the same asset if the economy is in bad times at time t. For the moment, we allow both $V_G(t)$ and $V_B(t)$ to depend on time.

As hinted, the reason the asset can have different values in good and bad times in the first place is that it pays different dividends in the two states of the world. Let π_G be the dividend in good times, and π_B the dividend in bad times, where we assume that both π_G and π_B are independent of time, and such that $\pi_G > \pi_B \geq 0$. That is, dividends are non-negative, constant, and higher in good times than bad (which is why we call them good and bad).

Finally, let r be the interest rate, which we also treat as constant and exogenous in this setting.

Now we want to determine the price of the asset at any given point in time and state of the world. To do that, we rely of the assumption that there can be no opportunity for so-called *arbitrage*. Here this simply means that the return to holding the asset must equal what investors would get if selling it. Suppose first that we are in good times at time t. The no-arbitrage condition then states that

$$rV_G(t) = \pi_G + \dot{V}_G(t) - b[V_G(t) - V_B(t)]. \tag{37}$$

Similarly, if we are in bad times at time t, the no-arbitrage condition implies

$$rV_B(t) = \pi_B + \dot{V}_B(t) + g[V_G(t) - V_B(t)]. \tag{38}$$

Section B of the appendix derives (37) using so-called *dynamic programming*; (38) can be derived analogously.

However, an easier way to understand (37) and (38) is to use intuitive reasoning. The left-hand side of (37), $rV_G(t)$, gives the return from selling the asset for $V_G(t)$ and putting the money in the bank earning the interest rate r. The right hand-side of (37) gives the return to holding the asset, which equals the sum of the dividend, π_G, and the expected change in the value of the asset, $\dot{V}_G(t) - b[V_G(t) - V_B(t)]$. The latter in turn consists of two terms: the change in the value while staying in good times, $\dot{V}_G(t)$; and the expected loss incurred if there is a transition to the bad state, $b[V_G(t) - V_B(t)]$.

The intuition behind (38) are analogous, but involves an expected gain in the wake of a transition from bad times to good, as captured by $g[V_G(t) - V_B(t)]$.

1.4.1 Steady state

Suppose now that the economy is in a steady state where the value of the asset in any state of the world is constant, i.e., $\dot{V}_G(t) = \dot{V}_G(t) = 0$. Let V_G and V_B (without any time arguments) denote steady-state values. Rewriting (37) and (38) gives

$$rV_G = \pi_G - b[V_G - V_B], \tag{39}$$

and

$$rV_B = \pi_B + g[V_G - V_B], \tag{40}$$

which can be solved to find V_G and V_B as functions of the underlying parameters of the model: the interest rate, r; the dividends in good and bad times, π_G and π_B; and the rates at which the economy transitions between these states, b and g. First write the difference between (39) and (40) as

$$r[V_G - V_B] = (\pi_G - \pi_B) - (b + g)[V_G - V_B], \tag{41}$$

which can be solved for $V_G - V_B$ to give

$$V_G - V_B = \frac{\pi_G - \pi_B}{r + b + g}. \tag{42}$$

Then substitute (42) into (39) and (40) to get

$$V_G = \frac{1}{r}\left[\pi_G - b\left(\frac{\pi_G - \pi_B}{r + b + g}\right)\right] = \frac{(r + g)\pi_G + b\pi_B}{r\,(r + b + g)}, \tag{43}$$

and

$$V_B = \frac{1}{r}\left[\pi_B + g\left(\frac{\pi_G - \pi_B}{r + b + g}\right)\right] = \frac{g\pi_G + (r + b)\pi_B}{r\,(r + b + g)}. \tag{44}$$

Note from (43) and (44) that the asset value in one state is connected to the dividend in the other. The asset value in good times (V_G) is increasing in its dividend in bad times (π_B), and the value of the asset in bad times (V_B) is increasing in its dividend in good times (π_G). This follows from the fact that the economy transitions between good and bad times. The greater are the transition rates (b and g), the more sensitive the asset prices are to changes in the dividends pertaining to the other state; e.g., V_G increases more in response to a rise in π_B if b is large.

If we close down differences in dividends between the two states (setting $\pi_G = \pi_B$), then the asset's value becomes state independent ($V_G = V_B$). If we close down transitions between the states (setting $b = g = 0$), then the asset's value will depend only on the dividend of the state it is in ($V_G = \pi_G/r$ and $V_B = \pi_B/r$), since it will stay in that state forever.

2 Growth models

This section looks at a couple of different models of economic growth. [More to be rewritten in due time...]

2.1 The Solow model

Two assumptions are central to the Solow model: a Neoclassical production function and a constant saving rate. The latter is the more controversial of the two assumptions, and arguably the main reason why the Solow model is not commonly used in the research community today. It means we cannot us the Solow model to study how saving (and investment) reacts endogenously in response to, say, a productivity shock. However, the Solow model is still a useful introduction to growth theory, and helpful to understand how other (more realistic) growth models work.

We begin with a summary of the model's notation, i.e., a description of what each variable represents. Similar to the presentation in Section 1.1, we let $K(t)$ denote the total capital stock of the economy, which is here a function of time, denoted t. Similarly, $Y(t)$ denotes total output at time t, and $L(t)$ the size of the labor force at time t. Then we let $A(t)$ denote labor-augmenting productivity at time t, where we refer to $A(t)L(t)$ as the *effective labor force*, or the *effective number of workers*.

2.1.1 Production

As mentioned, we assume a Neoclassical production function, which we denote by F. Just as in Section 1.1.1, it has two arguments, but these are now capital and *effective* labor, both of which depend on time. Specifically, output at time t is given by:

$$Y(t) = F\left(K(t), A(t)L(t)\right), \tag{45}$$

where F is assumed to satisfy Assumptions 1-5 in Section 1.1.1. Note $A(t)$ and $L(t)$ are not two separate arguments, but $A(t)L(t)$ is one single argument. In other words, the derivatives and limits referring to (time-independent) labor in Assumptions 1-5 in Section 1.1.1 here refer to effective labor, $A(t)L(t)$.

We can now define capital and output per effective worker as

$$k(t) = \frac{K(t)}{A(t)L(t)}, \tag{46}$$

and

$$y(t) = \frac{Y(t)}{A(t)L(t)}, \tag{47}$$

respectively. Since F exhibits CRS, we can use the insights from Section 1.1.2 to define the intensive-form production function as $f(k) = F(k, 1)$, and write output per effective worker as function of capital per effective worker:

$$y(t) = f(k(t)). \tag{48}$$

Since F is Neoclassical, we know that f satisfies the properties in (9)-(11), such as $f(0) = 0$, $f'(k) > 0$, and $f''(k) < 0$. This informs us about the shape of the graph of $f(k)$ in a diagram with k on the horizontal axis.

We do not have a good measure of $y(t)$, so we are usually more interested in $Y(t)$, which corresponds roughly to total GDP. Using (47) and (48) we see that

$$Y(t) = A(t)L(t)y(t) = A(t)L(t)f(k(t)). \tag{49}$$

We can also find an expression for *output per worker*, which we are going to denote by $x(t)$. Using (47) and (48) again, this can be written

$$x(t) = \frac{Y(t)}{L(t)} = A(t)y(t) = A(t)f(k(t)). \tag{50}$$

We can think of $x(t)$ as GDP per capita at time t, which is a common measure of economic development.

We are going to assume that $A(t)$ and $L(t)$ grow at constant and exogenous rates, g and n, respectively. That is,

$$\begin{aligned} \dot{A}(t) &= gA(t), \\ \dot{L}(t) &= nL(t). \end{aligned} \tag{51}$$

Recall from Section 1.3 that we can solve the differential equations in (51) to get $A(t) = A(0)e^{gt}$ and $L(t) = L(0)e^{nt}$.

The final assumption of the Solow model is a constant rate of saving. We let s denote the fraction of total output, $Y(t)$, that is saved at any given point in time. Since s is a fraction it holds that $0 \leq s \leq 1$. That s is constant means it does not depend on time, t, but this should not be taken too literally: we will pursue exercises where we let s make discrete jump from one constant level to another at some point in time.

2.1.2 Dynamics

The Solow model describes a closed economy. Therefore, total saving, $sY(t)$, equals total investment in new capital. Capital is assumed to depreciate over time at a rate δ. This means that the change in the capital stock at any given point in time equals

$$\dot{K}(t) = sY(t) - \delta K(t). \tag{52}$$

To interpret (52), consider an economy that starts off at time zero with a capital stock of $K(0)$. Suppose the economy does not invest at all in new capital, $s = 0$. Then the capital stock will evolve over time according to the differential equation $\dot{K}(t) = -\delta K(t)$, meaning that the capital stock at time t equals $K(t) = K(0)e^{-\delta t}$. That is, $K(t)$ approaches zero as t goes to infinity.

Before we start analyzing the model we are going to simplify the notation, by suppressing (dropping) the "(t)" part of the time-dependent variables. From now on, when we write k, y, $\dot{k}$, etc., we have to remember that these variables are actually functions of time. This way we can rewrite (46), (47), (48), (51), and (52) in the following more compact way:

$$\begin{aligned} k &= \tfrac{K}{AL}, \\ y &= \tfrac{Y}{AL}, \\ y &= f(k), \\ \dot{A} &= gA, \\ \dot{L} &= nL, \\ \dot{K} &= sY - \delta K. \end{aligned} \tag{53}$$

Using the expressions in (53), we want to find a differential equation that describes how capital per effective worker, k, evolves over time. This should be an expression for $\dot{k}$ in terms of k and things that are constant over time, namely the function f, and the parameters s, n, g, and δ.

To get there, some of the tricks we learned in Section 1.3 are helpful. We first log $k = K/(AL)$. This gives

$$\ln k = \ln K - \ln A - \ln L. \tag{54}$$

Differentiating both sides of (54) with respect to t gives

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n), \tag{55}$$

where we have used $\dot{A} = gA$ and $\dot{L} = nL$. Next, using $\dot{K} = sY - \delta K$ and the definitions of k and y, we can write $\dot{K}/K$ as

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = s\frac{y}{k} - \delta. \tag{56}$$

(Note that $y/k = [Y/(AL)]/[K/(AL)] = Y/K$.)

Finally, we can use $y = f(k)$, (55), and (56) to see that

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n+g+\delta), \tag{57}$$

or

$$\dot{k} = sf(k) - (n+g+\delta)k. \tag{58}$$

Since $sf(k)$ is investment per effective worker, we sometimes refer to $(n+g+\delta)k$ as "break-even" investment per effective worker. This is what actual investment per effective worker must equal for the capital stock per effective worker not to be falling over time, i.e., for $\dot{k}$ to be non-negative.

We can illustrate (58) in a diagram with k on the horizontal axis and the following three things on the vertical axis: output per effective worker, $f(k)$; (actual) investment per effective worker, $sf(k)$; and break-even investment per effective worker, $(n+g+\delta)k$. We will refer to this diagram as a break-even investment diagram.

We can read off $\dot{k}$ as the vertical distance between $sf(k)$ and $(n+g+\delta)k$. If $sf(k) > (n+g+\delta)k$, then $\dot{k} > 0$, meaning k is increasing over time. Vice versa, if $sf(k) < (n+g+\delta)k$, then $\dot{k} < 0$, meaning k is decreasing over time. This way, we can illustrate how k changes over time by drawing arrows along the horizontal axis, and quickly realize that k will converge over time to the level at which $\dot{k} = 0$.

2.1.3 Steady state

When $\dot{k} = 0$, we say that the economy is *in steady state*, or *on a balanced growth path*. The level of k at which $\dot{k} = 0$ is usually denoted k^*, and referred to as the steady-state level of k. In the break-even investment diagram, k^* is the point on the horizontal axis at which $sf(k)$ and $(n+g+\delta)k$ intersect.

Setting $\dot{k} = 0$ in (58) we see that k^* is defined from

$$sf(k^*) = (n+g+\delta)k^*. \tag{59}$$

The associated steady-state level of y, denoted y^*, is given by $y^* = f(k^*)$, which we can also read off the vertical axis in the break-even investment diagram.

Problem 9 *In the break-even investment diagram, show how k^* and y^* change in response to an increase in s from s_0 to s_1. Denote the associated levels of k^* and y^* before and after the change by k_0^* and k_1^*, and y_0^* and y_1^*, respectively.*

2.1.4 Time paths when starting off below steady state

We can also use the break-even investment diagram to figure out *time paths* of different variables. Consider an economy starting off at $t = 0$ with some capital stock per effective worker below its steady-state level: $k(0) < k^*$. At $t = 0$, it must hold that $\dot{k}(0) = sf(k(0)) - (n+g+\delta)k(0) > 0$. To verify this, we can just read off $\dot{k}(0)$ in the break-even investment diagram for $k(0) < k^*$.

If $k(0)$ is not too far below k^*, we can also see that $\dot{k}$ declines over time. However, $\dot{k}$ never reaches 0 in finite time, but rather approaches zero asymptotically as time goes to infinity, and k approaches its steady-state level k^*.

Since $\dot{k}$ is the slope of k, we can also figure out the time path of k from the path of $\dot{k}$. That is, k starts off at $t = 0$ below k^*, and then approaches k^* gradually over time. Note that k never intersects k^*, because the slope of k (i.e., $\dot{k}$) approaches zero as k approaches k^*.

Using a similar logic, we can draw the time paths of $\dot{k}/k$ (which is declining over time) and $\ln k$ (which is increasing over time with a slope that is declining).

We return to the topic of time paths later when we look at how some of these variables react to shocks to the exogenous parameters, such as s.

2.1.5 Consumption and the Golden Rule

The Solow model also makes predictions about consumption. Let C denote total consumption of the economy. Then $c = C/(AL)$ is consumption per effective worker. Since a fraction s of output is saved, and the remainder is consumed, it must hold that $C = (1-s)Y$, and thus

$$c = (1-s)y = (1-s)f(k). \tag{60}$$

We can read c in the break-even investment diagram as the vertical distance between $f(k)$ and $sf(k)$. Since k and y are constant in steady state, so is c, and just as we denoted the steady-state levels of k and y by k^* and y^*, respectively, we denote the steady-state level of c by c^*.

Imposing steady state on (60), and using the definition of k^* in (59), we now see that

$$c^* = f(k^*) - sf(k^*) = f(k^*) - (n+g+\delta)k^*. \tag{61}$$

This says that steady-state consumption equals what is produced, net of what must be saved for the economy to be in steady state, both of which are increasing functions of the steady-state capital stock (all expressed in per-effective-worker terms), which in turn is an increasing function of the saving rate, s.

From (59) and (61), it can be seen that $c^* = 0$ when $s = 0$, since in this case $k^* = 0$, and thus $c^* = f(k^*) = f(0) = 0$. In other words, an economy with no saving has zero capital in steady state, and thus no output from which to consume. It also can be seen that $c^* = 0$ when $s = 1$, since then $c^* = f(k^*) - f(k^*) = 0$. That is, an economy that saves 100% of output will also have zero consumption.

We can now ask a theoretically quite interesting question: what level of s maximizes c^*? We call this the Golden Rule level of s, and in these notes, it will be denoted by s_{GR}.

We can interpret s_{GR} from a diagram with $s \in [0,1]$ on the horizontal axis and c^* on the vertical. The graph of c^* starts at $c^* = 0$ for $s = 0$, peaks at $s = s_{GR}$, and reaches $c^* = 0$ at $s = 1$.

To find an expression for s_{GR} (or, rather, an expression that defines it) we first remember that k^* is an (increasing) function of s, implicitly defined by (59). We can then maximize

$c^* = f(k^*) - (n+g+\delta)k^*$ in (61) with respect to s. The first-order condition says that

$$\frac{\partial c^*}{\partial s} = \left[f'(k^*) - (n+g+\delta)\right]\frac{\partial k^*}{\partial s} = 0. \tag{62}$$

We leave this part as an exercise:

Problem 10 *Implicitly differentiate (59) to find an expression for $\partial k^*/\partial s$. Show that $\partial k^*/\partial s > 0$ if $sf'(k^*) < n+g+\delta$. Illustrate $sf'(k^*)$ and $n+g+\delta$ in a break-even investment diagram, and show that $sf'(k^*) < n+g+\delta$ must hold.*

Since $\partial k^*/\partial s > 0$, the first-order condition in (62) can only hold if $f'(k^*) = n+g+\delta$. Let k^*_{GR} be the k^* associated with $s = s_{GR}$, defined from

$$f'(k^*_{GR}) = n+g+\delta. \tag{63}$$

We can illustrate k^*_{GR} in a break-even investment diagram with two graphs: $f(k)$ and $(n+g+\delta)k$. There we can read $f'(k^*_{GR})$ as the slope of $f(k)$ at $k = k^*_{GR}$, and we can read $n+g+\delta$ as the slope of the (straight) break-even investment line, $(n+g+\delta)k$. Where these slopes are equal, we have k^*_{GR} (on the horizontal axis). This is where the vertical distance between $f(k)$ and $(n+g+\delta)k$ is maximized. This is not a coincidence: when we are in steady state ($k = k^*$), steady-state consumption equals that vertical distance, since $c^* = f(k^*) - (n+g+\delta)k^*$; see (61).

Now you can add a third graph to the figure, namely $s_{GR}f(k)$. By the definition of s_{GR}, this must be positioned such $s_{GR}f(k)$ intersects $(n+g+\delta)k$ exactly where $k = k^*_{GR}$. Intuitively, there are many possible levels of s. Each s generates its own k^*, which we find at the intersection between $sf(k)$ and $(n+g+\delta)k$. The Golden Rule level of s, which we call s_{GR}, is the particular level of s that makes k^* equal to k^*_{GR}.

Note that the Solow model does not predict that s equals to s_{GR}. There is no agent (and no government or social planner) who sets s. Rather, s is an exogenous variable in the Solow model, and we (who set up the model) can let it vary between 0 and 1. We shall see later that if we let agents choose saving to maximize a well-defined intertemporal utility function, then they will never save so much that c^* is above its peak.

Problem 11 *Assume Cobb-Douglas production, $f(k) = k^\alpha$. Find expressions for k^*, y^*, k^*_{GR}, and s_{GR}.*

2.1.6 Time paths following a shock to s

We next return to the topic of time paths, which we discussed already, but now to explore how the economy reacts to a change in s. Specifically, we are going to assume that s increases from some constant level s_0 to some higher level $s_1 > s_0$ at some point in time that we call $\hat{t}$. That is,

$$s = \begin{cases} s_0 & \text{if } t < \hat{t}, \\ s_1 & \text{if } t \geq \hat{t}. \end{cases} \tag{64}$$

We can first use a break-even investment diagram to analyze how steady-state outcomes change when s increases from s_0 to s_1. The steady-state level of k increases from k_0^* to k_1^*, with self-explanatory notation, defined from

$$\begin{aligned} s_0 f(k_0^*) &= (n+g+\delta)k_0^*, \\ s_1 f(k_1^*) &= (n+g+\delta)k_1^*. \end{aligned} \tag{65}$$

Similarly, the steady-state levels of y associated with s_0 and s_1, respectively, are defined from

$$\begin{aligned} y_0^* &= f(k_0^*), \\ y_1^* &= f(k_1^*). \end{aligned} \tag{66}$$

In this section it helps the presentation a little if we revert to letting time arguments be explicit, writing $k(t)$, $y(t)$, $\dot{k}(t)$ etc. This allows us to distinguish between levels at different points in time. (Note, however, that we do not write s as a function of t, even though it is here formally a discontinuous function of t.)

We assume that the economy is in a steady state from time zero up until $\hat{t}$ (i.e., when the shock hits) associated with the pre-shock saving rate, s_0. This means that capital per effective worker at time $\hat{t}$ equals its pre-shock level, k_0^*. That is, $k(t) = k_0^*$ for $t \in [0, \hat{t}]$. Moreover, if $k(t)$ is constant, it must hold that the change in $k(t)$ is zero. That is, $\dot{k}(t) = 0$ for $t \in [0, \hat{t})$ (but not for $t = \hat{t}$, as we shall soon see).

Consider now $\dot{k}(t)$ at the point of the shock, $\dot{k}(\hat{t})$.[1] Recall that $k(\hat{t}) = k_0^*$ and that $s = s_1$ at $t = \hat{t}$; see (64). Evaluating (58) at $t = \hat{t}$ now gives

$$\begin{aligned} \dot{k}(\hat{t}) &= s_1 f(k(\hat{t})) - (n+g+\delta)k(\hat{t}) \\ &= s_1 f(k_0^*) - (n+g+\delta)k_0^* \\ &= s_1 f(k_0^*) - s_0 f(k_0^*) \\ &= (s_1 - s_0)\, y_0^* > 0, \end{aligned} \tag{67}$$

where the second equality uses $k(\hat{t}) = k_0^*$, the third equality uses (65), the fourth equality uses (66), and the inequality follows from $s_1 > s_0$. In other words, $\dot{k}(t)$ jumps up from 0 to $(s_1 - s_0)\, y_0^* > 0$ at $t = \hat{t}$ (i.e., when the rate of saving increases from s_0 to s_1).

From that point on (after $\hat{t}$), the time path of $\dot{k}(t)$ declines over time, approaching zero as t goes to infinity. We see this in the break-even investment diagram, where we can read

[1] In exams I sometimes express this in words as "$\dot{k}$ right after $\hat{t}$." The reason is that the time path of $k(t)$ is kinked at $\hat{t}$, and it may not be obvious how we should interpret $\dot{k}(\hat{t})$ when we think of it as the slope of $k(t)$. What I mean is simply the level to which $\dot{k}(t)$ jumps at the point of the shock, $\hat{t}$.

$\dot{k}(t)$ after $\hat{t}$ as the gap between $s_1 f(k(t))$ and $(n+g+\delta)k(t)$, which shrinks as $k(t)$ grows from k_0^* to k_1^*.

To sum up, $\dot{k}(t) = 0$ for $t < \hat{t}$; then $\dot{k}(t)$ jumps up to $(s_1 - s_0)\, y_0^* > 0$ at $t = \hat{t}$; then $\dot{k}(t)$ declines over time, approaching zero as time goes to infinity, $\lim_{t\to\infty} \dot{k}(t) = 0$.

Since the path of $k(t)$ has a slope that equals $\dot{k}(t)$ it follows that $k(t) = k_0^*$ for $t \leq \hat{t}$. The path has a kink at $t = \hat{t}$, when the slope changes. From $\hat{t}$ and onwards $k(t)$ then increases gradually over time, with diminishing slope, approaching k_1^*.

Note that $k(t)$ does not make any discrete jump. That would imply that $\dot{k}(t)$ is infinite at the point of the jump, and we know from (67) that this is not the case. In other words, $k(t)$ must be very close to k_0^* right after $\hat{t}$.

Consider next the time path of the growth rate of $k(t)$, i.e., $\dot{k}(t)/k(t)$. This will follow a path qualitatively similar to that of $\dot{k}(t)$. That is, it starts at zero, since $\dot{k}(t) = 0$ for $t < \hat{t}$. Then $\dot{k}(t)/k(t)$ jumps to $(s_1 - s_0)\, f(k_0^*)/k_0^*$ at $t = \hat{t}$; to see this, note that $k(\hat{t}) = k_0^*$, and $\dot{k}(\hat{t}) = (s_1 - s_0)\, f(k_0^*)$; see (67). Then $\dot{k}(t)/k(t)$ declines over time and approaches zero as time goes to infinity, since the numerator approaches zero and the denominator approaches something positive, $\lim_{t\to\infty} \dot{k}(t)/k(t) = 0/k_1^* = 0$.

Once we have the time path of $\dot{k}(t)/k(t)$ we can also find the time path of $\dot{y}(t)/y(t)$. Logging $y(t) = f(k(t))$, and differentiating with respect to time, we find that

$$\begin{aligned}
\frac{\dot{y}(t)}{y(t)} &= \frac{\partial \ln[y(t)]}{\partial t} \\
&= \frac{\partial \ln[f(k(t))]}{\partial t} \\
&= \frac{\partial \ln[f(k(t))]}{\partial f(k(t))} \frac{\partial f(k(t))}{\partial k(t)} \frac{\partial k(t)}{\partial t} \\
&= \frac{f'(k(t))\dot{k}(t)}{f(k(t))} \\
&= \left[\frac{f'(k(t))k(t)}{f(k(t))}\right] \frac{\dot{k}(t)}{k(t)} \\
&= \alpha(k(t)) \frac{\dot{k}(t)}{k(t)},
\end{aligned} \tag{68}$$

where the third equality uses the chain rule, and the last equality defines the function

$$\alpha(k) = \frac{f'(k)k}{f(k)}. \tag{69}$$

The left-hand side of (13), and $f'(k) > 0$, together tell us that $\alpha(k) \in (0, 1)$.

This means that the time path of $\dot{y}(t)/y(t)$ resembles, but falls below, that of $\dot{k}(t)/k(t)$, and makes a smaller jump at $\hat{t}$. We see right away that $\dot{y}(t)/y(t) = 0$ for $t < \hat{t}$. Then it can be shown that $\dot{y}(t)/y(t)$ jumps up to $(s_1 - s_0)\, f'(k_0^*)$ at $t = \hat{t}$. To see this, note that $y(\hat{t}) = f(k(\hat{t})) = f(k_0^*) = y_0^*$, and that $\dot{k}(\hat{t}) = (s_1 - s_0)\, y_0^*$ [see (67)]; then use the fourth equality in (68). Finally, we see that $\dot{y}(t)/y(t)$ declines over time after $\hat{t}$, and approaches zero as time goes to infinity, i.e., $\lim_{t\to\infty} \dot{y}(t)/y(t) = 0$.[2]

[2]To verify that $\dot{y}(t)/y(t)$ is declining monotonically after $\hat{t}$, we can use the fourth equality in (68) and the

Recall that $\dot{y}(t)/y(t)$ is the growth rate of income per *effective* worker. Next we will look for the time path of the growth rate of income *per worker*, $\dot{x}(t)/x(t)$, where $x(t)$ is given by (50) and (recall) can be thought of as GDP per capita. To that end, we log (50) and differentiate with respect to t, using $\dot{A}(t)/A(t) = g$. This gives

$$\frac{\dot{x}(t)}{x(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{y}(t)}{y(t)} = g + \frac{\dot{y}(t)}{y(t)}. \tag{70}$$

That is, the time path of $\dot{x}(t)/x(t)$ is identical to that of $\dot{y}(t)/y(t)$, but shifted up by the constant term g. (We assume that $g > 0$.) That is, $\dot{x}(t)/x(t) = g$ for $t < \hat{t}$. Then $\dot{x}(t)/x(t)$ jumps up to $g + (s_1 - s_0) f'(k_0^*)$ at $t = \hat{t}$, and thereafter declines over time approaching g as time goes to infinity, $\lim_{t\to\infty} \dot{x}(t)/x(t) = g$.

In this model, the growth effects of changes in the rate of saving, s, are thus temporary. What drives growth in per-capita income in the long run is productivity growth, which is here assumed to equal the exogenous constant g. Therefore, in these types of models (where g is exogenous) differences in s across economies can only explain differences in *levels* of per-capita incomes, not differences in growth rates. Models that can explain differences in growth rates are called *endogenous growth models*, a topic we will return to later.

Having figured out the time path of $\dot{x}(t)/x(t)$ it is straightforward to find the time path of $\ln[x(t)]$, which has a slope equal to $\dot{x}(t)/x(t)$. That is, the path of $\ln[x(t)]$ has slope g for $t < \hat{t}$, then has a kink at $t = \hat{t}$, becoming temporarily steeper and converging back to the original slope of g.

Another way to understand the time paths of $\dot{x}(t)/x(t)$ and $\ln[x(t)]$ is to recall that $\dot{A}(t)/A(t) = g$ implies that $A(t) = A(0)e^{gt}$. Using (50) we then see that

$$\ln[x(t)] = \ln[A(t)y(t)] = \ln[A(t)] + \ln[y(t)] = \ln[A(0)] + gt + \ln[y(t)], \tag{71}$$

which shows that the slope of $\ln[x(t)]$, when plotted against time, t, equals $g + \dot{y}(t)/y(t)$.

The last time path we are going to draw is that of consumption per effective worker, $c(t) = (1 - s)y(t)$. This turns out to be the trickiest path of them all. Before $\hat{t}$ it holds that $s = s_0$ and $y(t) = y_0^*$, so $c(t) = (1 - s_0)y_0^* = c_0^*$ for $t < \hat{t}$. Then $c(t)$ drops down to $c(t) = (1 - s_1)y_0^* < c_0^*$ at $t = \hat{t}$; note that $y(t)$ does not jump, but s does.

After $\hat{t}$, $c(t) = (1 - s_1)y(t)$, which is increasing over time, since $y(t)$ is increasing and s_1 is constant. As time goes to infinity $c(t)$ approaches $(1 - s_1)y_1^* = c_1^*$.

Now comes the difficult question: will the new level to which $c(t)$ is converging in the long run be higher, or lower, than the initial (pre-shock) level? In other words, which is greater, c_0^* or c_1^*? We actually already answered this question in Section 2.1.5. It depends

fact that $k(t)$ is increasing over time. This implies that $f'(k(t))$ is decreasing over time [since $f''(k(t)) < 0$] and that $f(k(t))$ increasing over time.

on how s_0 and s_1 compare to the Golden Rule level of s, what we called s_{GR}. In two cases, we can provide a definite answer:

(1) If $s_0 < s_1 < s_{GR}$, then $c_1^* > c_0^*$. That is, an increase in s leads to higher consumption in the long run.

(2) If $s_1 > s_0 > s_{GR}$, then $c_1^* < c_0^*$. That is, an increase in s leads to lower consumption in the long run.

As mentioned in Section 2.1.5, the Solow model does not predict that s should end up at the Golden Rule level, or above or below it either. It makes no predictions about s, since s is treated as exogenous. However, one can argue that a level of s below the Golden Rule level is the more plausible case, in the sense that it corresponds most closely to a setting where rational agents choose their saving. We will see this when we look at the Ramsey model later.

To understand why, not that in Case (1) the rise in s is followed by a temporary fall in consumption at $t = \hat{t}$ [from c_0^* to $(1 - s_1)y_0^* < c_0^*$], but higher consumption on the long run. This could happen in a model with rational agents if they suddenly change their time preferences and become more patient. Case (2) amounts to increasing saving when it is already above the Golden Rule level. This implies that agents forego consumption in the short run without any long-run reward. If saving was kept constant at s_0, agents would have higher consumption at all points in time after $\hat{t}$, compared to when they increased saving to s_1. This could not happen if saving decisions were made by rational agents.

In other words, models in which saving is treated as exogenous can generate strange results. This is one reason why the Solow model is not used by researchers.

2.1.7 Steady-state income effects from changes in s

We are often interested in the quantitative implications of growth models. One example is how output levels depend on rates of saving. We observe large differences between countries in levels of GDP per capita, which is a common measure of standards of living. The order of magnitude is such that the richest countries may have 20-30 times as high GDP per capita levels as the poorest. Policy makers may thus be interested in whether they can raise GDP per capita levels in the long run by increasing levels of saving and investment, e.g., by more sound fiscal policies, or through credit market reforms.

Suppose for the moment that all countries have the same levels of labor productivity, $A(t)$, at all points in time [which amounts to assuming that $A(0)$ and g are the same across countries]. Then, according to the Solow model, differences in GDP per capita, what we have labelled $x(t) = A(t)y(t)$, are driven by differences in $y(t)$, i.e., y^* in steady state. I am not claiming that this is a plausible assumption, but since we have not yet looked at theories explaining what drives growth in $A(t)$, we can start off exploring this case first to see how far we get.

Consider thus a number of economies that are all in steady state, all with different levels of s, and associated levels of y^*. We want to know by how many percent y^* increases if we increase s by 1%. To answer this, we are going to derive an expression for the elasticity of y^* with respect to s. Using the notation in Section 1.2, we will denote this $\varepsilon_{y^*,s}$, which can be written

$$\varepsilon_{y^*,s} = \frac{\partial y^*}{\partial s}\frac{s}{y^*} = \frac{\partial f(k^*)}{\partial s}\frac{s}{f(k^*)}. \tag{72}$$

We want to find an expression for $\varepsilon_{y^*,s}$ in terms of something we can put numbers on. Here I follow Romer's book and derive an expression for $\varepsilon_{y^*,s}$ in terms of the steady-state capital share, $\alpha(k^*)$, where the function $\alpha(k)$ is defined in (69).

Conventional wisdom says that $\alpha(k^*)$ is around 1/3 for most countries, so if we can find an expression for $\varepsilon_{y^*,s}$ in terms of $\alpha(k^*)$ we can readily get a number for $\varepsilon_{y^*,s}$ that we can compare to data. So the task now is to find an expression for $\varepsilon_{y^*,s}$ in terms of $\alpha(k^*)$.

In the Cobb-Douglas case, this is straightforward. Then $\alpha(k^*)$ is constant, usually denoted by just α, which is then a parameter (rather than a function). In that case, it is easily seen that income per effective worker equals $y^* = (s/[n+g+\delta])^{\alpha/(1-\alpha)}$; see Problem 11. Using the insights from Section 1.2 we see right away that this gives an elasticity of $\varepsilon_{y^*,s} = \alpha/(1-\alpha)$.

Instead I will follow Romer's book, and consider the general case where $\alpha(k^*)$ depends on k^*. First we use the chain rule to note that

$$\frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*}{\partial s}. \tag{73}$$

From Problem 10 we recall

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)} = \frac{f(k^*)}{s\left[\frac{f(k^*)}{k^*} - f'(k^*)\right]}, \tag{74}$$

where we have used the definition of k^* in (59) to note that $n+g+\delta = sf(k^*)/k^*$. Multiplying (73) by $s/f(k^*)$, using (72) and (74), we now see that

$$\begin{aligned}\varepsilon_{y^*,s} &= \tfrac{\partial f(k^*)}{\partial s}\tfrac{s}{f(k^*)} \\ &= \frac{\frac{f'(k^*)k^*}{f(k^*)}}{1-\frac{f'(k^*)k^*}{f(k^*)}} \\ &= \tfrac{\alpha(k^*)}{1-\alpha(k^*)}.\end{aligned} \tag{75}$$

The result in (75) (which mimics that of the Cobb-Douglas case) tells us that with $\alpha(k^*) \approx 1/3$, the elasticity equals

$$\varepsilon_{y^*,s} \approx \frac{1}{2}. \tag{76}$$

That is, a 1% increase in s is associated with roughly a .5% increase in y^*, and a 100% increase in s (a doubling) is associated with a 50% increase in y^*. Rates of saving and investment are hard to measure, but usually they do not differ between rich and poor countries by terribly more than 100%. By contrast, the gap in GDP per capita between rich and poor countries is much larger than 50%, more on the order of 3000% (meaning the richest countries are maybe 30 times as rich as the poorest). In other words, the Solow model cannot really explain per-capita GDP gaps across countries, at least not with a capital share around one third.

One way to visualize this is to plot data on $\ln y^*$ against $\ln s$ for all countries in the world, and see what the slope looks like. Recall that that slope is the same as the elasticity, since $\varepsilon_{y^*,s} = \partial \ln y^* / \partial \ln s$; see (21). I have not compiled those data for these lecture notes, but if the reasoning above holds the slope of a simple regression line based on the data should be much larger than .5.

2.1.8 Speed of convergence

We can also get an idea about the Solow model's quantitative implications by calculating what we call the speed of convergence. Recall that an economy starting off below its steady state tends to grow over time as it approaches the steady state. It does not actually reach the steady state in finite time, but we can calculate approximately how log time it takes for the capital stock per effective worker to reach half-way to its steady-state level.

To that end we are going linearize the differential equation describing the evolution of k (where we again suppress the time argument when there is no risk of confusion). First define the right-hand side of (58) as the function $\phi(k)$, i.e.,

$$\dot{k} = sf(k) - (n + g + \delta)k = \phi(k). \tag{77}$$

We are going to first find a linearization—or, more precisely, a first-order Taylor approximation—of $\phi(k)$ about (close to) $k = k^*$. The first-order Taylor approximation says that

$$\phi(k) \approx \phi(k^*) + \phi'(k^*)\,(k - k^*)\,. \tag{78}$$

In words, this says that $\phi(k)$ is approximately equal to the sum of two terms: (1) the level of $\phi(k)$ at the point $k = k^*$; and (2) a correction term capturing how far k is from k^*, multiplied by how much $\phi(k)$ changes on the margin in response to changes in k when starting off at k^*.

In this case, we see right way from the definition of k^* in (58) that $\phi(k^*) = 0$. Next we see from (77) that

$$\phi'(k) = sf'(k) - (n + g + \delta), \tag{79}$$

which we can then evaluate at $k = k^*$ to find

$$\begin{aligned} \phi'(k^*) &= sf'(k^*) - (n + g + \delta) \\ &= \left[\tfrac{(n+g+\delta)k^*}{f(k^*)}\right] f'(k^*) - (n + g + \delta) \\ &= -(n + g + \delta)[1 - \alpha(k^*)], \end{aligned} \tag{80}$$

where the second equality uses (58) again to substitute for $s = (n + g + \delta)\, k^*/f(k^*)$, and the third equality uses the notation in (69).

Now (77) to (80) give us an (approximate) expression for $\dot{k}$ is terms of k, and things that are independent of time. This is a differential equation, which we can solve. To that end, we use some variable substitution. First, let $z = k - k^*$ and $\lambda = (n + g + \delta)[1 - \alpha(k^*)]$. Then note that $\dot{z} = \dot{k}$ (since k^* does not depend on time). Then we get $\dot{z} = -\lambda z$, which can be solved to give (after adding explicit time arguments) $z(t) = z(0)e^{-\lambda t}$. Reverting back to the original notation we get

$$k(t) - k^* = [k(0) - k^*]\, e^{-(n+g+\delta)[1-\alpha(k^*)]t}. \tag{81}$$

Note that $k(t) - k^*$ is the "gap" between the capital stock per effective worker at time t and the same variable in steady state; this would be a negative number if the economy approaches the steady state from below.

If we can put numbers on the variables in the exponent on the right-hand side of (81), then we can calculate how long it takes for the gap to close by half compared to $t = 0$. Suppose again that $\alpha(k^*)$ is 1/3. Then let $n = .01$, $g = .02$, and $\delta = .03$, so that

$$(n + g + \delta)[1 - \alpha(k^*)] = .06(2/3) = .04.$$

We think of these as annual rates. This means that population grows by 1% per year, productivity (and thus GDP on a balanced growth path) grows by 2% per year, and capital depreciates by 3% per year. These are plausible numbers, and in line with those in Romer's book.

Suppose now that gap has closed by half by $t = T$ compared to $t = 0$. The gap at time T is $k(T) - k^* = [k(0) - k^*]\, e^{-.04T}$, which we can set equal to $(1/2)\, [k(0) - k^*]$ to solve for T. This gives

$$e^{-.04T} = \frac{1}{2} = 2^{-1}, \tag{82}$$

or

$$T = \frac{\ln 2}{.04} \approx 17. \tag{83}$$

So according to the model it should take about 17 years for the gap to close by half. We interpret the answer in years because we set the values of n, g, and δ based on estimated annual rates.

2.1.9 Simulating the Solow model in continuous time

To simulate the Solow model, or any similar growth model, the most common approach is probably to set it up in discrete time, letting each period correspond to one year.

Here we will instead simulate an approximation of the same continuous-time version of the Solow model that we have already analyzed. Recall that the definition of a time derivative in (24) is based on the increment in the time argument going to zero. If we instead let that time increment be very small but strictly positive we get a good approximation of the time derivative. Let that small time increment be $\Delta > 0$. Then we can write $\dot{k}$ is as

$$\dot{k}(t) \approx \frac{k(t+\Delta) - k(t)}{\Delta}. \tag{84}$$

Then using (58), and pretending that the approximate equality is exact, we get

$$\frac{k(t+\Delta) - k(t)}{\Delta} = sf(k(t)) - (n+g+\delta)k(t), \tag{85}$$

or, if rearranging,

$$k(t+\Delta) = k(t) + \Delta\left[sf(k(t)) - (n+g+\delta)k(t)\right]. \tag{86}$$

Next we assume a parametric form for $f(k(t))$, e.g., Cobb-Douglas or CES. We can then make numerical assumptions about the parameters of $f(k(t))$, the start value for $k(t)$ [i.e., $k(0)$], and the remaining model parameters, s, n, g, and δ. Once we have that we can use (86) to simulate the model as follows: given $k(0)$ we compute $k(\Delta)$; given $k(\Delta)$ we compute $k(2\Delta)$; given $k(2\Delta)$ we compute $k(3\Delta)$; and so on.

Problem 12 *Consider a Solow model hit by a shock to s, as described in Section 2.1.6. You will be provided with a template Matlab script file that simulates this economy for a numerical example, assuming Cobb-Douglas production; link posted here. To that script file (and with the same parameter values) write command lines that do the following:*
(i) *Create a new plot, similar to Figure 2, which shows the time path of $\ln(x(t))$. The figure should include the same path as in Figure 2 and also a new path showing the pre-shock path, i.e., the path for $\ln(x(t))$ if s had stayed unchanged at s_0.*
(ii) *Create a new plot showing the path for consumption per effective worker, $c(t)$. How can*

we tell from the plot that we assumed $s_0 < s_1 < \alpha$?

(iii) *Create a new plot showing a normalized path for output per effective worker, equal to one before the shock. That is, the plot should show the path of a new variable defined as $\widetilde{y}(t) = y(t)/y_0^*$. Approximately how many percent greater is output per effective worker after 20 years compared to before the shock?*

(iv) *Figure 1 shows the time path of $y(t)$ and a few more things. Create a new plot just like that, and include one more line that is tangent to $y(t)$ at $t = \hat{t}$. Hint: use paper and pencil to find an expression for $\dot{y}(\hat{t})$.*

2.2 The Ramsey model

The Solow model assumed that saving was a constant and exogenous fraction of output. The Ramsey model instead assumes that agents choose saving (and consumption) to maximize an intertemporal utility function. You have most likely encountered utility functions before, and probably know that an intertemporal utility function is one that depends on consumption at different points in time.

We are first going to let $C(t)$ be consumption per worker at time t, so that $c(t) = C(t)/A(t)$ is consumption per effective worker. This notation is a little inconsistent, since for other variables we use capital letters to denote total amounts. For example, $K(t)$ is the total capital stock and $k(t) = K(t)/[A(t)L(t)]$ is capital per effective worker. Here I just follow Romer, who lets $C(t)$ be consumption per worker. This does not matter in the end because we will focus mostly on units per effective worker, but keep this in mind below.

2.2.1 Utility

The world described by the Ramsey model consists of H households. As in the Solow model, the total population (and labor force) at time t is $L(t)$, so each household has $L(t)/H$ members at time t.

Now we can write the (time-zero) utility of a household as

$$U = \int_0^\infty e^{-\rho t} u\left(C(t)\right) \frac{L(t)}{H} dt \tag{87}$$

where $u\left(C(t)\right)$ is the "instantaneous" utility of each worker (household member) from consumption at time t; $\rho > 0$ is the utility discount rate; and $L(t)/H$ is the number of household members at time t.

Utility discounting captures the idea that agents (workers) value consumption less the further into the future it takes place. That is, the weight assigned to utility at time t is $e^{-\rho t}$, which is declining with t. The weight declines more rapidly with time if ρ is large, so a low ρ

implies a more patient agent. We refer to $e^{-\rho t}$ as the discount factor, while ρ is the discount rate.

We assume that instantaneous utility function satisfies $u'(C) > 0$ and $u''(C) < 0$, implying positive marginal utility of consumption at any point in time, and decreasing marginal utility as consumption increases. To get nice analytical solutions we are (mostly) going to let instantaneous utility be given by the function

$$u(C) = \frac{C^{1-\theta}}{1-\theta}, \tag{88}$$

where $\theta > 0$ and $\theta \neq 1$ (which we must assume since we divide by $1-\theta$). Note that $u'(C) = C^{-\theta} > 0$ and $u''(C) = -\theta C^{-(1+\theta)} < 0$.

Another common instantaneous utility function is the logarithmic one, $u(C) = \ln(C)$. This in fact corresponds to $\theta = 1$ in (88). To see this, set $\theta = 1$ in the expression for the marginal utility derived from (88). This gives $u'(C) = C^{-1}$, which is the same as the derivative of $\ln(C)$.

The utility function in (88) isoelastic, since the elasticity of $u(C)$ with respect to C equals a constant $1-\theta < 1$. We sometimes also call this utility function CRRA, which stands for Constant Relative Risk Aversion. Risk aversion is defined as minus the elasticity of $u'(C)$ with respect to C, i.e., $-\left[\partial u'(C)/\partial C\right]\left[C/u'(C)\right] = -u''(C)C/u'(C)$, which equals θ with the utility function in (88) (and 1 with logarithmic utility). The term "risk aversion" can be a little misleading in this context, because there is no uncertainty in the model (i.e., there are no stochastic variables). Another way to formulate it is to say that the instantaneous utility function exhibits constant *intertemporal elasticity of substitution*, which here equals $1/\theta$. We return to this concept below.

2.2.2 Factor prices

A worker earns labor income $A(t)w(t)$ at time t, where (as in the Solow model) $A(t)$ is the level of labor productivity at time t, and $w(t)$ is here the wage per efficiency unit of labor, given by the marginal product to effective labor, which can be written

$$w(t) = f(k(t)) - f'(k(t))k(t). \tag{89}$$

As in the Solow model, f is the intensive-form production function and $k(t)$ is capital per effective worker. Note that we here define $w(t)$ differently than in Romer's end-of-chapter Problem 1.9, where $w(t)$ was the marginal product to labor, rather than effective labor. Again, this just follows Romer's book.

The interest rate at time t is given by

$$r(t) = f'(k(t)), \tag{90}$$

which again follows Romer's end-of-chapter Problem 1.9, except that we here set capital depreciation to zero, $\delta = 0$, as Romer does when presenting the Ramsey model.

2.2.3 Budget constraint

As in the Solow model, we also assume constant growth rates of $A(t)$ and $L(t)$, denoted g and n, respectively. Thus, $A(t) = A(0)e^{gt}$ and $L(t) = L(0)e^{nt}$.

Now we can write the intertemporal budget constraint for a household. In words, this should state that the household's initial wealth (at time zero), plus the present value of all its labor income at all points in time, equals (or exceeds) the present value of its consumption at all points in time. That is,

$$\int_0^\infty e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_0^\infty e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt. \tag{91}$$

where $R(t) = \int_0^t r(\tau) d\tau$, and where $e^{-R(t)}$ is the present-value factor discussed in Section 1.3.

To make sense of (91), consider first the term on the left-hand side. Each agent (household member) consumes $C(t)$ at time t, so the time-zero value of that agent's consumption at that point in time equals $e^{-R(t)}C(t)$. Multiplying by the total number of members in the household at that time, $L(t)/H$, gives the time-zero value of total consumption expenditures of the household at time t. Then integrating these discounted expenditures up from time zero to infinity gives us the sum of all discounted consumption expenditures for the eternity of the household (or dynasty).

The second term on the right-hand side of (91) sums up labor incomes in the same way, where we recall that each agent earns $A(t)w(t)$ at time t.

The first term on the right-hand side, $K(0)/H$, is the same as initial wealth owned by the household, which is the total initial capital stock divided by the number of households.

2.2.4 Rewriting utility function and budget constraints

The end task is to find a continuous time path for consumption that maximizes utility in (87) subject to the budget constraint in (91). To get there, we are first going to rewrite the both (87) and (91) in terms of consumption per effective worker, $c(t) = C(t)/A(t)$. Substituting $C(t) = A(t)c(t)$, $A(t) = A(0)e^{gt}$, and $L(t) = L(0)e^{nt}$ into (91), and then multiplying by H and and dividing by $A(0)L(0)$, gives

$$\int_0^\infty e^{(n+g)t - R(t)} c(t) dt \leq k(0) + \int_0^\infty e^{(n+g)t - R(t)} w(t) dt, \tag{92}$$

or

$$k(0) + \int_0^\infty e^{(n+g)t - R(t)} \left[w(t) - c(t) \right] dt \geq 0, \tag{93}$$

where we note that $k(0) = K(0)/[A(0)L(0)]$ is initial capital per effective worker.

To rewrite (87), we use (88) together with $C(t) = A(t)c(t)$ and $A(t) = A(0)e^{gt}$ to note that

$$[C(t)]^{1-\theta} = \left[c(t)A(0)e^{gt}\right]^{1-\theta} = [c(t)]^{1-\theta}\,[A(0)]^{1-\theta}\,e^{(1-\theta)gt}. \tag{94}$$

Now we can use (88), (94), and $L(t) = L(0)e^{nt}$ to rewrite (87) as

$$U = \frac{L(0)\,[A(0)]^{1-\theta}}{H}\int_0^\infty e^{-[\rho-(1-\theta)g-n]t}\left(\frac{[c(t)]^{1-\theta}}{1-\theta}\right)dt, \tag{95}$$

where we have collected all exponential factors on a common base, and then factored out things from the integral that do not depend on t. It helps to rewrite (95) as

$$U = B\int_0^\infty e^{-\beta t}\left(\frac{[c(t)]^{1-\theta}}{1-\theta}\right)dt, \tag{96}$$

where

$$B = \frac{L(0)\,[A(0)]^{1-\theta}}{H}, \tag{97}$$

and

$$\beta = \rho - (1-\theta)g - n = \rho - (n+g) + \theta g. \tag{98}$$

We can interpret β as the "effective" utility discount rate, which must be positive for utility in (96) to be finite. The reason is that $[c(t)]^{1-\theta}/(1-\theta)$ will be seen to converge to a non-zero constant, so unless $\beta > 0$ the integral would not be finite. Put another way, $e^{-\beta t}$ would not be declining over time, so the terms inside the integral would not converge to zero, making the integral "explode." We see that $\beta > 0$ requires

$$\rho > (1-\theta)g + n, \tag{99}$$

which we assume holds.

2.2.5 Lagrangian

The task now is to find a path for $c(t)$ that maximizes (96) subject to (92). In other words, we are choosing a whole function, rather than a single variable, or even a finite number of variables. However, we can think of this problem as choosing one level of $c(t)$ for every t, and pretend that the integrals are discrete sums. Then we get write a first-order condition that should hold at every point in time, t.

First set up this Lagrangian:

$$\mathcal{L} = B\int_0^\infty e^{-\beta t}\left(\frac{[c(t)]^{1-\theta}}{1-\theta}\right)dt + \lambda\left[k(0) + \int_0^\infty e^{(n+g)t-R(t)}\left(w(t)-c(t)\right)dt\right], \tag{100}$$

where λ is the Lagrangian multiplier. We want to differentiate $\mathcal{L}$ with respect to each $c(t)$, of which there are infinitely many, and then set each such derivative to zero. That is what the first-order condition for a utility maximum says. The trick is to think of the integrals as sums. When differentiating with respect to consumption at some t we treat consumption at all other points in time as additive constants, which just go away when we differentiate. Using this intuitive approach, Section A of the appendix derives $\partial U/\partial c(t)$ from a discrete approximation of U in (96). The same applies to the integral in the second terms of (100).

Thus, the first-order condition can be written

$$\frac{\partial \mathcal{L}}{\partial c(t)} = Be^{-\beta t}\left[c(t)\right]^{-\theta} - \lambda e^{(n+g)t-R(t)} = 0, \tag{101}$$

which must hold for all $t \geq 0$. The first term in (101) can be interpreted as the marginal utility of consumption at time t, discounted to time 0. Evaluating (101) at $t = 0$, we see that λ is the same as marginal utility of consumption at time zero, $B\left[c(0)\right]^{-\theta}$. Thus, the second term is the present value of capital at time t, evaluated at time-zero marginal utility by multiplying with $\lambda = B\left[c(0)\right]^{-\theta}$.

2.2.6 Euler equation

Since (101) holds for all $t \geq 0$ it implicitly defines the whole path of $c(t)$ in terms of t, $R(t)$, n, g, β, λ, and B. Instead of solving for $c(t)$ we are going to look for an expression for the growth rate of $c(t)$, i.e., $\dot{c}(t)/c(t)$.

First, recalling that $R(t) = \int_0^t r(\tau)d\tau$, we see that $\dot{R}(t) = r(t)$. That is, the derivative of a definite integral with respect to its upper limit equals the integrand, here $r(\tau)$, evaluated at that upper limit, here t; this is an application of Leibniz's rule.

Next, we use (101), and taking logs, we get

$$\ln B - \left[\rho - (n+g) + \theta g\right]t - \theta \ln\left[c(t)\right] = \ln \lambda + (n+g)t - R(t), \tag{102}$$

where we have used the expression for β in (98). Then taking the time derivative of both hand sides of (102), recalling that B and λ are constant, and that $\dot{R}(t) = r(t)$, we get

$$-\rho + (n+g) - \theta g - \theta\frac{\dot{c}(t)}{c(t)} = (n+g) - r(t) \tag{103}$$

which can be rearranged as

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[r(t) - \rho - \theta g\right]. \tag{104}$$

This is known as the *Euler equation*. (Note that the Euler equation derived here refers to the continuous-time Ramsey model; the same equation for the discrete-time version looks a little different.)

The Euler equation tells us that $c(t)$ is growing when $r(t) > \rho + \theta g$, and contracting when $r(t) < \rho + \theta g$. Intuitively, a high interest rate makes the agent want to push consumption into the future, since the return to foregone consumption today in terms of consumption tomorrow is higher.

How strongly the growth rate of $c(t)$ reacts to changes in $r(t)$ depends on θ. Recall that $1/\theta$ is the intertemporal elasticity of substitution. If θ is small (close to zero), and $1/\theta$ thus large, then the growth rate of $c(t)$ is more sensitive to changes in $r(t)$. Intuitively, a θ close to zero means the instantaneous utility function is close to linear, which implies a high elasticity of substitution between consumption today and consumption tomorrow.

We can also derive the Euler equation by first finding $c(t)$. Use $\lambda = B\left[c(0)\right]^{-\theta}$ and the expression for β in (98) to rewrite the condition in (101) as

$$Be^{-[\rho-(n+g)+\theta g]t}\left[c(t)\right]^{-\theta} = B\left[c(0)\right]^{-\theta} e^{(n+g)t-R(t)}, \tag{105}$$

Letting the factors B and $e^{(n+g)t}$ cancel, inverting both sides, and rearranging, we arrive at

$$\left[c(t)\right]^{\theta} = \left[c(0)\right]^{\theta} e^{[R(t)-(\rho+\theta g)t]}, \tag{106}$$

or

$$c(t) = c(0)e^{[R(t)-(\rho+\theta g)t]/\theta}. \tag{107}$$

Now we can log and differentiate (107), recalling $\dot{R}(t) = r(t)$, to get the Euler equation in (104). Alternatively, we can integrate (104) to find (107). The two are each other's equivalents.

2.2.7 Dynamical system

Using (90) to substitute for $r(t)$ in the Euler equation in (104) we get a differential equation that defines $\dot{c}(t)$ in terms of $c(t)$ and $k(t)$:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[f'(k(t)) - \rho - \theta g\right]. \tag{108}$$

[We can multiply both sides of (108) by $c(t)$ to isolate $\dot{c}(t)$ on the left-hand side, if we want.]

Since $\dot{c}(t)$ depends on another time-dependent variable than $c(t)$ itself, namely $k(t)$, we cannot easily see how $c(t)$ evolves over time, since the paths of $k(t)$, and thus also $f'(k(t))$, will depend on how $c(t)$ evolves. However, if we can find an equation for $\dot{k}(t)$ in terms of $c(t)$ and $k(t)$ (and nothing else involving t), then we have two equations that characterize the *joint* dynamics of $c(t)$ and $k(t)$.

The task is thus to find an expression for $\dot{k}(t)$ in terms of $c(t)$ and $k(t)$. Recalling that we set capital depreciation to zero ($\delta = 0$) we know that the change in the total capital stock, $\dot{K}(t)$, must equal total investment, which in turn is given by total output minus total consumption. Total output can be written as the effective labor forces times output per effective worker, $Y(t) = A(t)L(t)f(k(t))$. Total consumption equals consumption per worker times the number of worker, $L(t)C(t) = A(t)L(t)c(t)$; recall the peculiar notation in Romer's book where $C(t)$ is consumption per worker and $c(t)$ is consumption per effective worker. This gives

$$\begin{aligned}\dot{K}(t) &= Y(t) - L(t)C(t) \\ &= A(t)L(t)f(k(t)) - A(t)L(t)c(t) \\ &= A(t)L(t)\left[f(k(t)) - c(t)\right].\end{aligned} \tag{109}$$

Logging and differentiating $k(t) = K(t)/[A(t)L(t)]$ it follows that

$$\begin{aligned}\tfrac{\dot{k}(t)}{k(t)} &= \tfrac{\dot{K}(t)}{K(t)} - \left[\tfrac{\dot{A}(t)}{A(t)} + \tfrac{\dot{L}(t)}{L(t)}\right] \\ &= \tfrac{A(t)L(t)}{K(t)}\left[f(k(t)) - c(t)\right] - (g+n) \\ &= \tfrac{1}{k(t)}\left[f(k(t)) - c(t)\right] - (g+n),\end{aligned} \tag{110}$$

where the second equality uses (109), $\dot{A}(t)A(t) = g$, and $\dot{L}(t)/L(t) = n$, and the third equality uses $k(t) = K(t)/[A(t)L(t)]$. Now we can multiply (110) by $k(t)$ to get

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)\,k(t). \tag{111}$$

This corresponds to (58) in the Solow model, except that we here assume $\delta = 0$. The rate of saving—which is still the fraction of income not consumed—is now endogenous and time dependent, and can be written

$$s(t) = \frac{f(k(t)) - c(t)}{f(k(t))} = \frac{\dot{k}(t) + (n+g)\,k(t)}{f(k(t))}. \tag{112}$$

Now (108) and (111) together constitute a *two-dimensional system of differential equations*, or a *two-dimensional dynamical system*. We can rewrite them together as

$$\begin{aligned}\tfrac{\dot{c}(t)}{c(t)} &= \tfrac{1}{\theta}\left[f'(k(t)) - \rho - \theta g\right], \\ \dot{k}(t) &= f(k(t)) - c(t) - (n+g)\,k(t).\end{aligned} \tag{113}$$

Problem 13 *Using the same notation and logic as in Section 2.1.9 for the Solow model, where Δ was a small positive time increment, apply the equations in (113) to derive approximate expressions for $c(t+\Delta)$ and $k(t+\Delta)$. The answers should be in terms of $c(t)$, $k(t)$, the function f and its derivatives, Δ, and exogenous parameters.*

2.2.8 Phase diagram

To illustrate the time of paths of $c(t)$ and $k(t)$ we usually draw a *phase diagram*, which has $k(t)$ on the horizontal axis and $c(t)$ on the vertical. A coordinate in that diagram shows the levels of $c(t)$ and $k(t)$ at any given point in time. Those levels in turn determine the signs (and sizes) of $\dot{c}(t)$ and $\dot{k}(t)$, which tell us whether $c(t)$ and $k(t)$ are increasing, decreasing, or constant. Changes in $c(t)$ and $k(t)$ over time are represented as movements along the axes where the two variables are measured.

First we are going to find the so-called *loci* (plural of locus), meaning curves or lines, along which $c(t)$ and $k(t)$, respectively, are constant. From there we can figure out how these variables change off the same loci. This can be represented by arrows pointing up or down, or left or right; I sometimes say “north or south” and “west or east.” Using those arrows we can read off a unique path in the phase diagram for any initial position.

The ($\dot{c} = 0$)-locus The ($\dot{c} = 0$)-locus gives us combinations of $c(t)$ and $k(t)$ along which $\dot{c}(t) = 0$. We see right away from (113) that $\dot{c}(t) = 0$ when $k(t) = k^*$, where k^* is defined from

$$f'(k^*) = \rho + \theta g. \tag{114}$$

That is, regardless of the level of $c(t)$, it holds that $\dot{c}(t) = 0$ as long as $k(t) = k^*$. Moreover, because $f''(k) < 0$, it must hold that $\dot{c}(t) > 0$ when $k(t) < k^*$, since then $f'(k(t)) > f'(k^*) = \rho + \theta g$; vice versa, $\dot{c}(t) < 0$ when $k(t) > k^*$, since then $f'(k(t)) < f'(k^*) = \rho + \theta g$. We can see this in a diagram with k on the horizontal axis, by drawing $f'(k)$ (with negative slope) and $\rho + \theta g$ (which is constant and does not depend on k). Where they intersect you have k^*, and you can read the sign of $\dot{c}(t)$ to the left and right of k^*.

Now we can draw the ($\dot{c} = 0$)-locus in the phase diagram as a vertical line at $k(t) = k^*$. To the left (or west) of that locus, we know that $\dot{c}(t) > 0$, which can be illustrated with arrows pointing up (or north), since we measure $c(t)$ on the vertical axis. To the right (or east) of the locus, it follows that $\dot{c}(t) < 0$, illustrated by arrows pointing down (or south).

The ($\dot{k} = 0$)-locus The ($\dot{k} = 0$)-locus gives us combinations of $c(t)$ and $k(t)$ along which $\dot{k}(t) = 0$. Using (113) we see that $\dot{k}(t) = 0$ when $c(t) = \Psi(k(t))$, where the function $\Psi(k)$ is defined from

$$\Psi(k) = f(k) - (n + g)\, k. \tag{115}$$

That is, combinations of $c(t)$ and $k(t)$ such that $c(t) = \Psi(k(t))$ give us the ($\dot{k} = 0$)-locus. To draw the ($\dot{k} = 0$)-locus in the phase diagram, first note that

$$\Psi'(k) = f'(k) - (n + g)\,, \tag{116}$$

and

$$\Psi''(k) = f''(k) < 0. \tag{117}$$

This means that $\Psi(k)$ reaches a maximum at $k = k^*_{GR}$, where $\Psi'(k^*_{GR}) = 0$, or

$$f'(k^*_{GR}) = n + g. \tag{118}$$

This is the same k^*_{GR} that we encountered in (63) when we analyzed the Solow model. The only difference is that we have assumed $\delta = 0$ when we set up the Ramsey model, so we end up with $n + g$ on the right-hand side of (118).

Finally, we see that $\Psi(0) = f(0) = 0$.

We can now draw the $(\dot{k} = 0)$-locus in the phase diagram. The locus starts at the origin, has positive slope until $k = k^*_{GR}$, where it peaks, after which point the slope turns negative.

We have deduced that $\dot{k}(t) = 0$ when $c(t) = \Psi(k(t))$. From (113), and the definition of $\Psi(k)$ in (115), we can also figure out the sign of $\dot{k}(t)$ when we are off the locus: if $c(t) < \Psi(k(t))$, then $\dot{k}(t) > 0$; if $c(t) > \Psi(k(t))$, then $\dot{k}(t) < 0$. This can be illustrated with "east-west" arrows in the phase diagram at positions off the $(\dot{k} = 0)$-locus.

2.2.9 Steady state and the saddle path

We can now illustrate the $(\dot{c} = 0)$-locus and the $(\dot{k} = 0)$-locus in the same phase diagram. The steady-state levels of $c(t)$ and $k(t)$ in the Ramsey model are found where the two loci intersect. At that point, $\dot{c}(t) = \dot{k}(t) = 0$, so both $c(t)$ and $k(t)$ are constant over time. We learned already that $\dot{c}(t) = 0$ requires $k(t) = k^*$, so k^* is the steady-state level of $k(t)$. Then the steady level of $c(t)$ is given by

$$c^* = f(k^*) - (n + g)\, k^* = \Psi(k^*). \tag{119}$$

When we draw the phase diagram we should note that the $(\dot{c} = 0)$-locus is positioned to the left (west) of the peak of the $(\dot{k} = 0)$-locus, i.e., $k^* < k^*_{GR}$, as shown in this problem:

Problem 14 *Draw the graph of $f'(k)$ in a diagram with k on the horizontal axis. Use (114) and (118) to show that the assumption in (99) implies $k^* < k^*_{GR}$.*

This insight relates to what we found in Section 2.1.5 when we analyzed steady-state consumption in the Solow model. Since saving is set exogenously in the Solow model it can generate the peculiar result that the capital stock ends up above its Golden Rule level in steady state, meaning we could increase consumption for all future by permanently reducing saving. In the Ramsey model, where rational agents choose saving, this cannot happen. This hinges on $\rho > (1 - \theta)g + n$, meaning the discount rate is large enough to ensure that utility is bounded. In other words, as long as agents discount the future enough for utility to be finite in the first place, the economy will never accumulate a capital stock above the Golden Rule level.

The steady-state rate of saving in the Ramsey model can be found by setting $\dot{k}(t) = 0$ and $k(t) = k^*$ in (112), which gives

$$s^* = \frac{f(k^*) - c^*}{f(k^*)} = \frac{(n+g)k^*}{f(k^*)}. \tag{120}$$

Problem 15 *Find c^*, k^*, y^*, s^*, and k^*_{GR} when the production function is Cobb-Douglas, $f(k) = k^\alpha$.*

We can also read the dynamic paths of $c(t)$ and $k(t)$ from the phase diagram. The loci divide the space into four segments with motion arrows pointing in different directions: northeast, northwest, southwest, and northwest. If we forget about the economic interpretations for a moment, we can think of the phase diagram as a map of a sea where currents take little rafts in different directions.

Suppose, for example, that the raft is dropped off at time zero a little west of the ($\dot{c} = 0$)-locus and a little south of the ($\dot{k} = 0$)-locus, i.e., $k(0) < k^*$ and $c(0) < \Psi(k(0))$. Then the raft will drift with the currents toward the northeast. At some point in time it is likely to intersect one of the loci. Suppose it intersects the ($\dot{k} = 0$)-locus; at that precise point in time it drifts straight northward, since there is no east-west current when it is exactly on the ($\dot{k} = 0$)-locus. A moment later it has entered the region to the north of the ($\dot{k} = 0$)-locus, while still to the west of the ($\dot{c} = 0$)-locus. Here the currents will carry the raft to the northwest. Eventually it intersects the vertical axis, at which point $k(t) = 0$.

The motion arrows would then appear to drag the raft further north along the vertical axis, but since $c(t)$ cannot exceed $f(k(t))$ we know that $c(t)$ must drop to zero from the time when $k(t) = 0$.

Had we dropped the raft off at the same point on the horizontal axis, i.e., the same $k(0)$, but a little bit further south, with a lower $c(0)$, then the currents would pull it in a more easterly direction where it eventually intersects the ($\dot{c} = 0$)-locus, and later the horizontal axis.

We can now see that, for a given level of $k(0)$, there exists a unique level of $c(0)$ such that the trajectory of the economy leads to the steady state, i.e., $k(t)$ approaches k^* and $c(t)$ approaches c^*. This is called the *saddle path* and it never intersects any of the axis.

Moreover, if the household (or dynasty) is infinitely lived, then the budget constraint in (93) says that the economy must be on the saddle path. This is known as the *transversality condition.*

2.2.10 Finite time horizon

To see where the transversality condition comes from, we can consider the optimal path if the household had a finite time horizon. In that case, the upper limit of the integrals in (100) would be some finite T, say, rather than to infinity. That is, T would be the point in time when the world ends. The first-order condition in (101) looks the same, and we can still derive the same Euler equation, but it would only apply to $t \leq T$. Since consumption would not generate any utility after T, it would be optimal for the household to have zero capital at time T, i.e., $k(T) = 0$. Given some initial level of capital, $k(0) < k^*$, the household would set initial consumption, $c(0)$, to be on a trajectory such that $k(T) = 0$.

We can illustrate this trajectory in the phase diagram, similar to the raft example above. First $c(t)$ and $k(t)$ increase, then the trajectory intersects the $(\dot{k} = 0)$-locus, at which point $k(t)$ reaches a maximum and then starts to decrease, while $c(t)$ continues to increase. The point in time when the trajectory intersects the vertical axis is when the world ends, T. We can draw the associated time path of $k(t)$ in a diagram with t on the horizontal axis, with $k(t)$ first increasing, then decreasing, and finally reaching zero at $t = T$.

The transversality condition essentially states that, if T is infinite, then the optimal path should be such that $k(t)$ never goes to zero.

2.2.11 Time paths following a shock to ρ

In the Solow model we looked at the time paths of the economy when it was hit by a shock to the exogenous saving rate, s. We can explore the effects of a similar shock in the Ramsey model, namely a fall in the discount rate, ρ, some point in time, $\hat{t}$. Remember that a high ρ implies less weight on the future, which translates to lower saving, so an increase in s in the Solow model corresponds to a fall in ρ in the Ramsey model.

We see from (114), and the fact that $f''(k) < 0$, that a fall in ρ from ρ_0 to ρ_1 is associated with a rise in k^* from k_0^* to k_1^*. That is, a fall in ρ shifts out the $(\dot{c} = 0)$-locus in the phase diagram. The $(\dot{k} = 0)$-locus, which was defined by the function $\Psi(k)$ in (115), does not involve ρ, so it stays unchanged. However, the new steady state is read off at the new intersection between the two loci, so we see that the fall in ρ from ρ_0 to ρ_1 is associated with a rise in c^* from c_0^* to c_1^*.

We have figured what happens in steady state. To understand how the economy reacts at $\hat{t}$ (the point in time when ρ falls), we assume that it was initially in a steady state associated with the higher (pre-shock) level of ρ. Note that $k(t)$ cannot make discrete jumps. The transversality condition requires that the economy must be on the saddle path. Therefore, $c(t)$ jumps down at time $\hat{t}$ to the new saddle path that leads to the new steady state. This means that $c(t)$ falls, while $k(t)$ and $f(k(t))$ stay unchanged, implying a rise in the rate of saving, as seen from (112).

After the shock, we can read the paths of $c(t)$ and $k(t)$ from the phase diagram as the economy converges to the new steady state. That is, $k(t)$ and $c(t)$ grow and asymptotically approach their new steady state levels, k_1^* and c_1^*.

2.2.12 Simulating the Ramsey model in continuous time

To simulate time paths of different variables in the Ramsey model, we can apply the approximations derived in Problem (13), and follow the same approach as for the Solow model in Section 2.1.9.

We first make assumptions about a production function (e.g., Cobb-Douglas), parameter values, and initial values for $c(t)$ and $k(t)$, i.e., $c(0)$ and $k(0)$.

Recall that the paths of $c(t)$ and $k(t)$ will crash into one of the axes if we are not on the saddle path. To allow for such paths in our simulations, we also add some commands which ensure that $k(t)$ never becomes negative, and that $c(t) = 0$ whenever $k(t) = 0$. With these adjustments, the solution for $k(t + \Delta)$ in Problem (13) can be written

$$k(t + \Delta) = \max\{0, k(t) + \Delta\,[f(k(t)) - c(t) - (n + g)k(t)]\}. \tag{121}$$

The corresponding expression for $c(t + \Delta)$ can be written

$$c(t + \Delta) = \begin{cases} c(t) + \frac{c(t)\Delta}{\theta}\,[f'(k(t)) - (\rho + \theta g)] & \text{if } k(t + \Delta) > 0, \\ 0 & \text{if } k(t + \Delta) = 0, \end{cases} \tag{122}$$

where the second line imposes the constraint that $c(t) = 0$ whenever $k(t) = 0$.

We can then iterate on the (121) and (122) in Matlab. One example is given in the template script file posted here. There we assume a Cobb-Douglas form for $f(k(t))$, with $\alpha = 1/3$. We set $\theta = 1$ and $\rho = .02$, keeping all other parameters as in the Solow model simulation. Start values for $k(t)$ and $c(t)$ are set to $k(0) = .25k^*$ and $c(0) = .5c^*$, which generates a path where $k(t)$ hits zero after about 28 years.

Problem 16 *Keeping all else unchanged in the template code (posted here), adjust $c(0)$ so that the economy starts off roughly on the saddle path. That is, set $c(0)$ so that $k(t)$ and $c(t)$ are still growing and approaching their steady state values throughout the whole simulation. You should do it through trial and error, and not calculate any exact value for $c(0)$, but you should be able to figure out in what direction to move $c(0)$. [Hint: two decimal places is enough.] Once you have found a value that seems to work, try doubling the number of years for the simulation from $T = 100$ to $T = 200$. Does it still look like you were on the saddle path?*

2.3 Endogenous growth models

In the versions of the Solow and Ramsey models presented so far, output per effective worker, $y(t)$, converges to a non-growing steady-state level, y^*. This implies that output per worker, $A(t)y(t)$, grows at the same rate as $A(t)$, i.e., g. This rate is exogenously given, so the models do not really explain what drives long-run growth in output per capita. That is, we cannot change any other exogenous parameters of the model to study how g responds. Now we are going to look at models where the variable g is endogenous.

2.3.1 The AK model

The easiest way to generate endogenous growth is to drop the assumption about a diminishing marginal product of capital, $\lim_{K\to\infty} \partial F(K,L)/\partial K = 0$. Consider the Solow model set up in Section 2.1, but now let output be given by a so-called AK production function, here written as

$$Y(t) = ZK(t). \tag{123}$$

where $Z > 0$ is an exogenous and constant productivity parameter. This can be interpreted as Cobb-Douglas production function with a capital share of one, $\alpha = 1$.

We would usually denote the parameter Z by A—which is why it is called an AK production function—but since we have earlier used $A(t)$ denote the time-dependent labor-augmenting productivity we here call it Z to avoid confusion.

Moreover, we now let labor-augmenting productivity be constant and equal to one. That is, $g = 0$ and $A(0) = 1$, meaning $A(t) = A(0)e^{gt} = 1$ for all t. This serves to close down the channel through which we were able to generate sustained growth in output per worker earlier. It follows that capital per effective worker is now the same as capital per worker, $k(t) = K(t)/[A(t)L(t)] = K(t)/L(t)$. Similarly, output per effective worker, $y(t)$, is now the same as output per worker, which we have denoted by $x(t)$ earlier. That is, $y(t) = Y(t)/[A(t)L(t)] = Y(t)/L(t) = x(t)$.

We are interested in the growth rate of $x(t) = Y(t)/L(t)$. With $L(t)$ growing at rate n we quickly see that

$$\frac{\dot{x}(t)}{x(t)} = \frac{\dot{Y}(t)}{Y(t)} - n = \frac{\dot{K}(t)}{K(t)} - n, \tag{124}$$

where the last equality uses the fact that Z in (123) is constant, meaning $Y(t)$ and $K(t)$ grow at the same rate. Using (52) and (123) we get $\dot{K}(t) = sZK(t) - \delta K(t)$, or

$$\frac{\dot{K}(t)}{K(t)} = sZ - \delta, \tag{125}$$

which is the growth rate of total capital (and total output). Together (124) and (125) now give

$$\frac{\dot{x}(t)}{x(t)} = sZ - \delta - n. \tag{126}$$

We have thus been able to derive an expression for the growth rate of output per worker, which in this model is determined endogenously as function of the exogenous variables s, Z, n, and δ. If $sZ > \delta + n$, then the growth rate of positive. We also see that a rise in s here generates a faster growth in perpetuity, rather than just a short-run effect, as was the case in the Solow model with exogenous growth.

Problem 17 *Consider the Ramsey model Section 2.2, but with the same AK technology as in (123), and with $A(t) = 1$ for all t (and thus $g = 0$). Suppose the rate of saving, $s(t)$, is constant in steady state (i.e., on a balanced-growth path). Show that this implies that $c(t)$ and $y(t)$ must grow at the same rate. Applying the Euler equation, find an expression for that growth rate in terms of Z, ρ and θ.*

The AK model generates endogenous growth by dropping the assumption about a Neoclassical production function. This has a couple of strange implications, at least if we take it literally. For example, it means that the marginal product of labor is zero. Also, the marginal product of capital does not decline with the existing stock of capital, implying that real interest rates do not decline as an economy grows, and do not differ between countries at different stages of development (as long as they have the same Z).

There are other ways to generate endogenous growth that are similar to the AK model, but do not share these problems, e.g., letting human capital be an input in the production function. We are not going to look at those models here, but instead explore a so-called two-sector model.

2.3.2 A two-sector model

The two-sector model we are going to set up next resembles the Solow model we analyzed in Section 2.1, but it interprets $A(t)$ as something that can be produced, e.g., knowledge, technology, or "ideas." Here we are going to call it knowledge.

There are two sectors in this economy: one produces goods, and the other produces new knowledge. As before, $K(t)$ and $L(t)$ denote the total stocks of capital and labor, and $A(t)$ is labor-augmenting productivity, which is here the same as the stock of knowledge. This captures the notion that more knowledge makes workers more productive.

Capital and labor is split between the two sectors. We let the shares of $K(t)$ and $L(t)$ used in knowledge production be a_K and a_L, respectively, which are constant and exogenous, and such that $a_K \in (0, 1)$ and $a_L \in (0, 1)$ (since they are shares).

By contrast, knowledge is non-rivalrous, meaning both sectors can use the whole stock of $A(t)$.

Goods output is denoted $Y(t)$ as before. We assume that the goods sector uses a Cobb-Douglas production function with capital share $\alpha \in (0,1)$, meaning goods output at time t equals

$$Y(t) = [(1-a_K)K(t)]^{\alpha} [(1-a_L)A(t)L(t)]^{1-\alpha}, \tag{127}$$

where $(1-a_K)K(t)$ and $(1-a_L)L(t)$ are the inputs of capital and labor in goods production, since the shares a_K and a_L are used in knowledge production.

The knowledge sector produces *new* knowledge, which adds to the stock of *existing* knowledge. Since we think of $A(t)$ as the stock of existing knowledge, it follows that $\dot{A}(t)$ equals new knowledge. The production function for the knowledge sector has three inputs—capital, labor, and existing knowledge—as given by this parametric form:

$$\dot{A}(t) = B\,[a_K K(t)]^{\beta} [A(t)]^{\theta} [a_L L(t)]^{\gamma}, \tag{128}$$

where $B > 0$ is a productivity parameter, and where $\gamma > 0$, $\beta > 0$, $\theta > 0$, and

$$\beta + \theta < 1. \tag{129}$$

The assumption in (129) means that the production function for new knowledge exhibits decreasing returns to scale with respect to the two factors that are accumulable, $K(t)$ and $A(t)$. This serves to rule out explosive growth, as we shall see later.

Accumulation of $K(t)$ is given by the expression in (52), except that we here assume zero capital depreciation, $\delta = 0$. Using (127) we get

$$\dot{K}(t) = sY(t) = s\,[(1-a_K)K(t)]^{\alpha} [(1-a_L)A(t)L(t)]^{1-\alpha}. \tag{130}$$

Finally, we assume that $L(t)$ grows at rate $n > 0$, just as before. It will be seen to matter that $n > 0$.

We are looking for expressions for the growth rates of $K(t)$ and $A(t)$, which we denote $g_K(t) = \dot{K}(t)/K(t)$ and $g_A(t) = \dot{A}(t)/A(t)$, respectively. Using (130) we can write $g_K(t)$ as

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha} \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}. \tag{131}$$

Using (128) we can write $g_A(t)$ as

$$g_A(t) = \frac{\dot{A}(t)}{A(t)} = Ba_K^{\beta} a_L^{\gamma} [K(t)]^{\beta} [A(t)]^{\theta-1} [L(t)]^{\gamma}. \tag{132}$$

To simplify, we can define the constants

$$\begin{aligned} c_K &= s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}, \\ c_A &= Ba_K^{\beta} a_L^{\gamma}, \end{aligned} \tag{133}$$

to write (131) and (132) as

$$g_K(t) = c_K \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}, \tag{134}$$

and

$$g_A(t) = c_A \left[K(t)\right]^{\beta} \left[A(t)\right]^{\theta-1} \left[L(t)\right]^{\gamma}. \tag{135}$$

Note that c_K and c_A do not denote consumption. Rather, they are constant factors containing the time-independent variables that affect growth rates.

We are looking for steady-state levels of $g_K(t)$ and $g_A(t)$. To get there, we want to find expressions for $\dot{g}_K(t)$ and $\dot{g}_A(t)$ that we can set equal to zero. Easiest is to first derive expressions for "growth rates in growth rates," i.e., $\dot{g}_K(t)/g_K(t)$ and $\dot{g}_A(t)/g_A(t)$. Logging (134), and then differentiating with respect to time, gives

$$\begin{aligned} \tfrac{\dot{g}_K(t)}{g_K(t)} &= (1-\alpha)\left[\tfrac{\dot{A}(t)}{A(t)} + \tfrac{\dot{L}(t)}{L(t)} - \tfrac{\dot{K}(t)}{K(t)}\right] \\ &= (1-\alpha)\left[g_A(t) + n - g_K(t)\right], \end{aligned} \tag{136}$$

where we have used $g_K(t) = \dot{K}(t)/K(t)$ and $g_A(t) = \dot{A}(t)/A(t)$, and recalled the assumption that $\dot{L}(t)/L(t) = n$. In the same manner, logging (135), and then differentiating with respect to time, gives

$$\begin{aligned} \tfrac{\dot{g}_A(t)}{g_A(t)} &= \beta\tfrac{\dot{K}(t)}{K(t)} + (\theta-1)\tfrac{\dot{A}(t)}{A(t)} + \gamma\tfrac{\dot{L}(t)}{L(t)} \\ &= \beta g_K(t) + (\theta-1)\,g_A(t) + \gamma n. \end{aligned} \tag{137}$$

The steady state growth rates of $K(t)$ and $A(t)$ are denoted g_K^* and g_A^*. We can derive expressions for these by setting $\dot{g}_K(t) = 0$, $\dot{g}_A(t) = 0$, $g_K(t) = g_K^*$ and $g_A(t) = g_A^*$ in (136) and (137). From (136) we get

$$g_K^* = n + g_A^*, \tag{138}$$

and from (137) we get

$$\beta g_K^* = (1-\theta)g_A^* - \gamma n. \tag{139}$$

To solve for g_K^* and g_A^*, we can first substitute (138) into (139) to get

$$\beta g_K^* = \beta(n + g_A^*) = (1-\theta)g_A^* - \gamma n, \tag{140}$$

which gives $(1-\theta-\beta)g_A^* = (\beta+\gamma)\,n$, or

$$g_A^* = \frac{(\beta+\gamma)n}{1-\theta-\beta}. \tag{141}$$

Then substituting (141) into (138) we get

$$g_K^* = n + g_A^* = \left(1 + \frac{\beta+\gamma}{1-\theta-\beta}\right)n = \left(\frac{1-\theta+\gamma}{1-\theta-\beta}\right)n. \tag{142}$$

We have now derived expressions for the endogenous steady-state growth rates of $K(t)$ and $A(t)$ in terms of the exogenous variables β, γ, θ, and n.

We can also find the steady-state growth rate of total output per capita, which we can denote g_Y^*. From (127) we see that

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha) \left[\frac{\dot{A}(t)}{A(t)} + n \right], \tag{143}$$

which in steady state becomes

$$\begin{aligned} g_Y^* &= \alpha g_K^* + (1-\alpha)(g_A^* + n) \\ &= (g_A^* + n) + \alpha (g_K^* - g_A^* - n) \\ &= g_A^* + n, \end{aligned} \tag{144}$$

where the third equality uses (138).

Then recall that output per capita, $x(t) = Y(t)/L(t)$, grows at the rate $g_x(t) = \dot{Y}(t)/Y(t) - n$, which in steady state becomes

$$g_x^* = g_Y^* - n = g_A^* = \frac{(\beta+\gamma)n}{1-\theta-\beta}, \tag{145}$$

where the second equality uses (144) and the third uses (141). This tells us that the growth rate of output per capita equals the steady-state growth rate of $A(t)$, just as in the Solow model with exogenous growth. The news here is that growth rate of $A(t)$ is determined endogenously as a function of other model variables.

We see that a higher n leads to a higher g_x^* in the model. The reason is that labor is an input in the production of knowledge, which determines labor productivity. The model thus seems to predict that growth in GDP per capita should be higher in countries with faster population growth. This is not quite what we see across countries today, where slow rates of growth in GDP per capita tends to be associated with high rates of population growth. However, there is some evidence from the preindustrial era of economic development that population growth has exerted a positive effect on growth in technology (although not living standards).

We also note that some parameters that some model parameters play no role for the growth rate of output per capita in this model. For example, the rate of investment, s, does not show up in (145), like in did in the corresponding equation for the AK model; see (126).

Dynamics So far we have only looked at the steady-state levels of $g_K(t)$ and $g_A(t)$. We are also interested in whether the economy will converge to this steady state, and how the paths of $g_K(t)$ and $g_A(t)$ can look in that transition. To the end, we can draw a phase diagram, like we did for the Ramsey model in Section 2.2. This will be a diagram with $g_A(t)$ on the horizontal axis, and $g_K(t)$ on the vertical, and two loci along which $g_K(t)$ and $g_A(t)$ are constant (meaning their time derivatives are zero). These loci intersect at the steady state.

The $(\dot{g}_K = 0)$-locus can be found by setting (136) to zero, which produces

$$g_K(t) = g_A(t) + n. \tag{146}$$

This gives combinations of $g_K(t)$ and $g_A(t)$ such that $g_K(t)$ is constant, meaning $\dot{g}_K(t) = 0$. We also see from (136) that $g_K(t) > g_A(t) + n$ implies $\dot{g}_K(t) < 0$, and that $g_K(t) < g_A(t) + n$ implies $\dot{g}_K(t) > 0$. We can now draw (146) in a phase diagram, and indicate with motion arrows how $g_K(t)$ evolves off the locus. That is, we move north and south at different sides of the locus, since we measure $g_K(t)$ on the vertical axis.

Similarly, setting (137) to zero gives is the $(\dot{g}_A = 0)$-locus as

$$g_K(t) = \left(\frac{1-\theta}{\beta}\right) g_A(t) - \frac{\gamma n}{\beta}, \tag{147}$$

which we can also draw in a phase diagram. Note that $\dot{g}_A(t) > 0$ when $g_K(t)$ exceeds the right-hand side of (147), and that $\dot{g}_A(t) < 0$ when it falls below. We can indicate this with motion arrows pointing east and west on different sides of the locus, since we measure $g_A(t)$ on the horizontal axis.

The $(\dot{g}_K = 0)$-locus has an intersect of $n > 0$ and a slope of one. The $(\dot{g}_A = 0)$-locus has a negative intercept, $-\gamma n/\beta < 0$, and a slope steeper than one, since (129) implies $(1-\theta)/\beta > 1$. It follows that the two loci must intersect, and where they intersect we find g_K^* and g_A^*.

The two loci divide the phase diagram into four segments, with motion arrows pointing southeast, southwest, northwest, and northeast. We can start off anywhere in the diagram, at some coordinate $g_K(0)$ and $g_A(0)$, and follow the motion arrows to find the path. Regardless of how we set $g_K(0)$ and $g_A(0)$ the path always leads to the steady state, which means the dynamic system is globally stable. However, the trajectory may intersect one of the loci, implying that $\dot{g}_K(t)$ or $\dot{g}_A(t)$ changes sign along the transition.

3 Growth accounting

We have learned from, e.g., the Solow model with exogenous growth that sustained growth in incomes per capita requires sustained growth in labor-augmenting productivity, what we labelled $A(t)$. While this might seem realistic enough, we do not have any concrete measures of $A(t)$, in either levels or growth rates. However, we can apply a Neoclassical production function with labor-augmenting productivity growth, like that in Section 2.1.1, to "back out" the *contribution* of growth in $A(t)$ to growth in total output.

We start with a general intensive-form production function, $y(t) = f(k(t))$, where lower-case letters denote units per effective worker. Using the notation in (49), we can write total output as

$$Y(t) = A(t)L(t)y(t) = A(t)L(t)f(k(t)). \tag{148}$$

Logging and differentiating with respect to time gives us an expression for growth in total output:

$$\begin{aligned}
\frac{\dot{Y}(t)}{Y(t)} &= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} + \frac{\dot{y}(t)}{y(t)} \\
&= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} + \alpha(k(t))\frac{\dot{k}(t)}{k(t)} \\
&= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} + \alpha(k(t))\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}\right] \\
&= [1-\alpha(k(t))]\left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right] + \alpha(k(t))\frac{\dot{K}(t)}{K(t)},
\end{aligned} \tag{149}$$

where we have used (68) and $k(t) = K(t)/[A(t)L(t)]$. In words, this says that growth in total output equals the sum of two terms: (1) the labor share of output times growth in productivity-augmented labor; and (2) the capital share of output times the growth rate of the capital stock. We can solve (149) for $[1-\alpha(k(t))]\,\dot{A}(t)/A(t)$ to write:

$$[1-\alpha(k(t))]\,\frac{\dot{A}(t)}{A(t)} = \frac{\dot{Y}(t)}{Y(t)} - \alpha(k(t))\frac{\dot{K}(t)}{K(t)} - [1-\alpha(k(t))]\,\frac{\dot{L}(t)}{L(t)}. \tag{150}$$

The left-hand side of (150) is a measure of how much productivity growth contributes to total output growth, sometimes called the *Solow residual*. The right-hand side is expressed in terms of things that we can measure, at least in principle: $\dot{Y}(t)/Y(t)$ corresponds to total growth in GDP; $\dot{L}(t)/L(t)$ corresponds to growth in the labor force or in the total number of hours worked; $\dot{K}(t)/K(t)$ corresponds to growth in the total capital stock, which can be decently estimated from observed levels of investment and realistic assumptions about the rate of capital depreciation; and $\alpha(k(t))$ can be calculated as the fraction of total income that is paid to capital owners. In other words, the Solow residual is the difference between observed output growth and the contribution made to output growth by these observable factors.

There is a large (and by now old) literature using this type of approach to explore, e.g., how much of the growth in East Asian countries can be attributed to productivity growth, as opposed to other factors (surprisingly little, some find). More recently, researchers have compared changes in the Solow residual to observable changes in the use of new technologies, e.g., internet technology and computers. Yet another related application is to use output data from several sectors of the economy to assess if existing production factors are allocated efficiently.

4 Labor market models

4.1 The Shapiro-Stiglitz model

The Shapiro-Stiglitz model is a labor market model where unemployment can arise in equilibrium due to asymmetric information. The basic idea is that workers may shirk, meaning not performing their work tasks. If they do, they become unproductive, so employers would like fire all shirking workers. Asymmetric information means the employer is not able to perfectly monitor workers, i.e., she does not know who is shirking, and who is not. However, shirkers are caught at some exogenous rate. Therefore, if the wage is sufficiently high, workers choose not to shirk, because being better paid means that they are also more keen on keeping their jobs.

The upshot is that the employer can induce workers not to shirk by paying them a high enough wage, what we call the no-shirking wage. Moreover, at that no-shirking wage, more workers are willing to work than firms are willing to hire. In that way, asymmetric information can give rise to unemployment.

The Shapiro-Stiglitz model is dynamic, with a few stochastic elements, involving transitions in and out of employment, and detection of shirking workers. In that way, it resembles the asset pricing model described in Section 1.4.

In any given period a worker can be in either one of three different states: employed and exerting effort, employed and shirking, and unemployed. Employed workers, whether shirking or not, earn a wage w. This wage will be determined in equilibrium later, but is taken as given by the workers.

Each state brings an associated *utility flow* to the worker, corresponding to the dividends in the model set up in the previous section. A worker who is employed and exerting effort gets a utility flow of $w - \bar{e}$, where (recall) w is the wage and $\bar{e}$ is the cost of exerting effort, measured in the same utility units as the wage. A worker who is employed and shirking gets the full utility flow of w, because she does not pay the cost of effort. An unemployed worker gets a utility flow of zero.

Workers transition between these three states at different hazard rates. Conditional on being employed and not shirking, the worker loses the job through so-called job separation at a rate $b > 0$, which is an exogenous parameter in this model.

If the worker is employed and shirking, she faces the risk of losing the job both through regular job separation at rate b, and now also when caught shirking, which happens at the rate $q > 0$, which we also treat as exogenous. Thus, a shirking worker transitions into unemployment at the rate $b + q$, which is higher than than of non-shirking workers.

Finally, unemployed workers transition into employment at a rate a, which will be determined endogenously later by equalizing the flows in and out of employment.

Each state has an associated *value*, defined as the discounted integral of all future expected utility flows. The rate at which these future utilities are discounted is denoted ρ, which corresponds to the interest rate r in the previous section. The value of being employed and not shirking is denoted by V_E; the value of being employed and shirking is denoted by V_S; and the value of being unemployed is denoted by V_U. These values relate to the utility flows and the discount rate in the same way that asset prices related to dividends and the interest rate in Section 1.4. Analogous to (39) and (40), we can write these three equations:

$$\rho V_E = w - \bar{e} - b\left(V_E - V_U\right), \tag{151}$$

$$\rho V_S = w - (b + q)\left(V_S - V_U\right), \tag{152}$$

and

$$\rho V_U = a\left(V_E - V_U\right). \tag{153}$$

Using the asset-dividend analogy, (151) states that the return to selling the asset and putting the money in the bank (the left-hand side) must equal the expected return if holding on to the asset (the right-hand side); the latter equals the "dividend" (i.e., $w - \bar{e}$), minus the expected loss in asset value, associated with going from a state of (non-shirking) employment to unemployment.

In (153) we make the assumption that workers who find jobs choose not to become shirkers, thus making a gain of $V_E - V_U$ at rate a (rather than $V_S - V_U$). This assumption is not important because workers will be indifferent between shirking and not shirking in equilibrium.

Comparing (151) and (152), we see that employed workers face a trade-off: shirking generates a higher dividend (higher utility flow) than not shirking, but also a higher rate of transition into unemployment.

To the employer it is optimal to set the wage so that workers choose not to shirk, but no higher; this wage is the no-shirking wage, which we shall denote by w^{NS}. In other words, the employer sets w high enough to ensure that $V_E \geq V_S$, but no higher, making the inequality hold with equality. This means that w^{NS} is defined as the level of w that makes $V_E = V_S$.

We can now use (151), (152), (153), and the condition $V_E = V_S$, to find the no-shirking wage, w^{NS}. First, taking the difference between (153) and (151) gives

$$\rho\left(V_E - V_U\right) = w - \bar{e} - (a + b)\left(V_E - V_U\right) \tag{154}$$

which we can solve for $V_E - V_U$ to get:

$$V_E - V_U = \frac{w - \bar{e}}{\rho + a + b}. \tag{155}$$

Then, setting $\rho V_E = \rho V_S$ in (152), and equalizing the right hand-sides of (151) and (152) (recalling $V_E = V_S$), gives

$$w - \bar{e} - b\left(V_E - V_U\right) = w - \left(b + q\right)\left(V_E - V_U\right), \tag{156}$$

which can be solved for $V_E - V_U$ to give:

$$V_E - V_U = \frac{\bar{e}}{q}. \tag{157}$$

Now we can use (155) and (157) to solve for w. This gives us this no-shirking wage as follows:

$$w^{\text{NS}} = \bar{e} + \frac{\bar{e}}{q}(\rho + a + b). \tag{158}$$

Intuitively, w^{NS} is increasing in $\bar{e}$, since with a higher utility cost of exerting effort employers must pay workers more to induce them not to shirk. We also see that w^{NS} is increasing in a. If unemployed workers find new jobs faster, then employed workers are more tempted to shirk, so employers must pay them more not to shirk. Also, w^{NS} is increasing in b, since a higher job separation rate (absent shirking) makes jobs less valuable; this makes workers less desperate to hold on to jobs, so employers must pay them more to induce them not to shirk.

Finally, we see that w^{NS} is decreasing in q, the rate at which shirkers are detected. Better monitoring reduces the need for wages as an instrument to keep workers from shirking. As $q \to \infty$, monitoring of workers becomes perfect, and the no-shirking wage approaches the cost of effort, $\bar{e}$. That is, with perfect monitoring, the worker's utility flow when working is the same as when unemployed.

To close the model, we are going to link the flows in and out of employment, as determined by the hazard rates a and b. Let the size of the total labor force be denoted by $\overline{L}$, which is exogenous and constant. The number of workers being employed is L^D, which denotes labor demand. That means that there are $\overline{L} - L^D$ workers who are unemployed at any point in time. At any point in time, $a(\overline{L} - L^D)$ unemployed workers flow into employment, and bL^D employed workers flow into unemployment. (Recall that no workers are shirking, so the rate at which they lose jobs equals b.) For employment to be constant, flows in and out of employment must equalize. Thus, $a(\overline{L} - L^D) = bL^D$, or

$$a = \frac{bL^D}{\overline{L} - L^D}. \tag{159}$$

Substituting (159) into (158) we can write the no-shirking wage as

$$w^{\text{NS}} = \bar{e} + \frac{\bar{e}}{q}\left(\rho + \frac{bL^D}{\overline{L} - L^D} + b\right) = \bar{e} + \frac{\bar{e}}{q}\left(\rho + \frac{b\overline{L}}{\overline{L} - L^D}\right). \tag{160}$$

Figure 1 illustrates the relationship between the no-shirking wage, w^{NS}, and labor demand, L^D. Note that $w^{\text{NS}} \to \infty$ as $L^D \to \overline{L}$, which means that there must be some unemployment in equilibrium for workers to be willing to exert effort at a finite wage. Note also the vertical intercept, as derived by setting $L^D = 0$ in (160).

We have also added a labor demand curve into Figure 1, showing how many workers the firms are willing to hire at different wages. The labor demand curve is downward sloping, and can be derived from a production function, equalizing the marginal product of labor to the wage (we do not need to derive it here).

As $q \to \infty$, and monitoring becomes perfect, the no-shirking curve approaches a standard labor-supply curve, as indicated by L^S in the diagram. That is,

$$L^S = \begin{cases} \overline{L} & \text{if} \quad w \geq \overline{e}, \\ \\ 0 & \text{if} \quad w < \overline{e}. \end{cases} \tag{161}$$

Point A in Figure 1 shows the equilibrium in a model without asymmetric information, i.e., when monitoring if perfect, as given by the intersection of L^D and L^S. As drawn here, we would then have full employment, meaning $L^D = \overline{L}$. (This is referred to as a Walrasian equilibrium in Romer's book.)

Point B shows the equilibrium under imperfect monitoring. As seen, the wage is higher and employment lower (unemployment higher) compared to the equilibrium under perfect monitoring (point A).

Why is point B the equilibrium? Why would a worker not be able to convince an employer to offer her a job at a wage slightly lower than the equilibrium wage at point B? Clearly, there are many unemployed workers who are willing to work at a lower wage, since $w > \overline{e}$ in equilibrium. The reason is that the employer knows that any worker paid less than the no-shirking wage would choose to shirk, and thus be unproductive, making it pointless to hire her at all.

APPENDIX

A Differentiating U in the Ramsey model

First approximate U as follows:

$$\begin{aligned} U &= B \int_0^\infty e^{-\beta t} \left(\frac{[c(t)]^{1-\theta}}{1-\theta} \right) dt \\ &\approx B \left[\frac{[c(0)]^{1-\theta}}{1-\theta} + e^{-\beta\Delta} \frac{[c(\Delta)]^{1-\theta}}{1-\theta} + e^{-2\beta\Delta} \frac{[c(2\Delta)]^{1-\theta}}{1-\theta} + ... \right] \\ &= B \sum_{t \in N} e^{-\beta t} \frac{[c(t)]^{1-\theta}}{1-\theta}, \end{aligned} \tag{A}$$

where N is the set of discrete points in time at which we approximate the integral, i.e.,

$$N = \{0, \Delta, 2\Delta, 3\Delta, ...\}. \tag{B}$$

The derivative of U with respect to $c(t)$, where t is any of the points in the sequence N, can now be approximated by

$$\frac{\partial U}{\partial c(t)} = Be^{-\beta t} [c(t)]^{-\theta}. \tag{C}$$

For example, $\partial U / \partial c(\Delta) = Be^{-\beta\Delta} [c(\Delta)]^{-\theta}$, $\partial U / \partial c(2\Delta) = Be^{-2\beta\Delta} [c(2\Delta)]^{-\theta}$, and so on. We can make Δ arbitrarily small, so we can think of $Be^{-\beta t} [c(t)]^{-\theta}$ as a continuous function of t.

B Dynamic programming

We are now going to derive an expression for (37); the derivation of (38) is analogous. To that end we use (an informal approach to) dynamic programming. The idea is to first consider a short time interval, here denoted by $\Delta > 0$. (In Romer's book and many other applications you will see "Δt" where I write Δ. This means the same thing. That is, you should think of Δt as *one* variable, not the product of two variables.) We are then going to assume (or pretend) that the economy does not transition between states over this time interval. This is approximately true if Δ is "short" ($\Delta \approx 0$), and becomes exactly true in the limit as Δ approaches zero ($\Delta \to 0$).

Suppose that we are in good times at time t. First we are going to specify the probability of a transition to bad times over the time interval Δ. With a Poisson process, that probability equals

$$1 - e^{-b\Delta}. \tag{D}$$

We now see that b is not quite the same as a probability, in the sense that it can be greater than one; the expression for the probability itself in (D) is always between zero and one, since $\Delta > 0$ and $b > 0$, which holds also if $b > 1$.

It is easy to see that $1 - e^{-b\Delta}$ is increasing in Δ. That is, the longer is the time interval, Δ, the more likely it is that the event (of transitioning from good times to bad) will have happened at the end of the time interval, at time $t + \Delta$. Put another way, the greater is Δ, the less likely it is that the economy will remain in good times. (Since the probability of transitioning from good times to bad equals $1 - e^{-b\Delta}$, the probability of staying in good times is just $e^{-b\Delta}$.)

We also see that $1-e^{-b\Delta}$ is increasing in b, holding constant Δ. The greater is the hazard rate, b, at which the economy transitions from good times to bad, the more likely the event is to have happened over any given time interval.

Similarly, if the economy is in bad times at time t, then the probability of it having transitioned to good times over a time interval of length $\Delta > 0$ equals

$$1 - e^{-g\Delta}. \tag{E}$$

Now let us talk about the value of a financial asset in these two states of the world. Given that the economy is in good times at time t, what will the expected value of the asset be at time $t+\Delta$? Note that the asset value at time $t+\Delta$ is random, because the economy may have either transitioned to bad times, or stayed in good times. In principle, the economy could have transitioned from good times to bad and back again, perhaps several times over, but we ignore that possibility here. Intuitively, the time interval is sufficiently "short" ($\Delta \approx 0$) for only one type of transition to have happened.

So we have only two types of outcome to consider—a transition to bad times, or no transition—which happen with probabilities $1 - e^{-b\Delta}$ and $e^{-b\Delta}$, respectively. If a transition to bad times took place, the asset value at the end of the interval equals $V_B(t + \Delta)$, and if it did not take place the value equals $V_G(t + \Delta)$. So if we start in good times at time t, then the expected asset value at time $t + \Delta$ equals

$$\left[1 - e^{-b\Delta}\right] V_B(t + \Delta) + e^{-b\Delta} V_G(t + \Delta). \tag{F}$$

Next, we can express $V_G(t)$ in terms of the expected value of the asset at $t + \Delta$, as given in (F). Being in good times at time t, and assuming no transition will have happened before the interval has ended, the owner of the asset gets a dividend $\pi_G\Delta$ over the time interval. At the end of time interval, the asset has an expected value as given by (F). Because this value accrues a little bit into the future it needs to be discounted. The discount rate is the same as the interest rate, r, so the discount factor equals $e^{-r\Delta}$. That is, an asset value of X at time $t + \Delta$ is worth $Xe^{-r\Delta}$ at time t. We can now write the value of the asset, if we are in good times at time t, as follows (with explanatory notes added):

$$V_G(t) = \underbrace{\pi_G\Delta}_{\text{dividend}} + \underbrace{e^{-r\Delta}}_{\text{discount factor}} \left\{ \underbrace{\left[1 - e^{-b\Delta}\right] V_B(t + \Delta) + e^{-b\Delta} V_G(t + \Delta)}_{\text{expected value to } t+\Delta} \right\}. \tag{G}$$

Although this looks messy, it should make intuitive sense. To derive (37) from (G), we are going to use some algebra and rules about limits and derivatives. First, we deduct $V_G(t+\Delta)$ from both sides of (G), multiply the factor $e^{-r\Delta}$ into the curly brackets expression, and then divide by Δ. This gives:

$$\frac{V_G(t) - V_G(t+\Delta)}{\Delta} = \pi_G + e^{-r\Delta}\left(\frac{1-e^{-b\Delta}}{\Delta}\right)V_B(t+\Delta) + \left(\frac{e^{-[b+r]\Delta}-1}{\Delta}\right)V_G(t+\Delta). \quad \text{(H)}$$

Next we let Δ approach zero. We see that the left-hand side of (H) becomes

$$\lim_{\Delta\to 0}\frac{V_G(t) - V_G(t+\Delta)}{\Delta} = -\lim_{\Delta\to 0}\frac{V_G(t+\Delta) - V_G(t)}{\Delta} = -\dot{V}_G(t), \quad \text{(I)}$$

where we have used the definition of a (time) derivative in (24). Looking at the expressions on the right-hand side that involve Δ, we see that some limits are trivial to find:

$$\lim_{\Delta\to 0} e^{-r\Delta} = 1, \quad \text{(J)}$$

$$\lim_{\Delta\to 0} V_B(t+\Delta) = V_B(t), \quad \text{(K)}$$

and

$$\lim_{\Delta\to 0} V_G(t+\Delta) = V_G(t). \quad \text{(L)}$$

To solve the two remaining expressions on the right-hand side of (H) involving Δ, we use l'Hôpital's rule, to get:

$$\lim_{\Delta\to 0}\frac{1-e^{-b\Delta}}{\Delta} = \lim_{\Delta\to 0}\frac{-(-b)e^{-b\Delta}}{1} = b, \quad \text{(M)}$$

and

$$\lim_{\Delta\to 0}\frac{e^{-[b+r]\Delta}-1}{\Delta} = \lim_{\Delta\to 0}\frac{-(b+r)e^{-b\Delta}}{1} = -(b+r). \quad \text{(N)}$$

Using (I) to (N), and letting Δ approach zero (H), we get

$$-\dot{V}_G(t) = \pi_G + 1 \times b \times V_B(t) - (b+r) \times V_G(t). \quad \text{(O)}$$

Rearranging (O) now gives the expression in (37).

C Figures

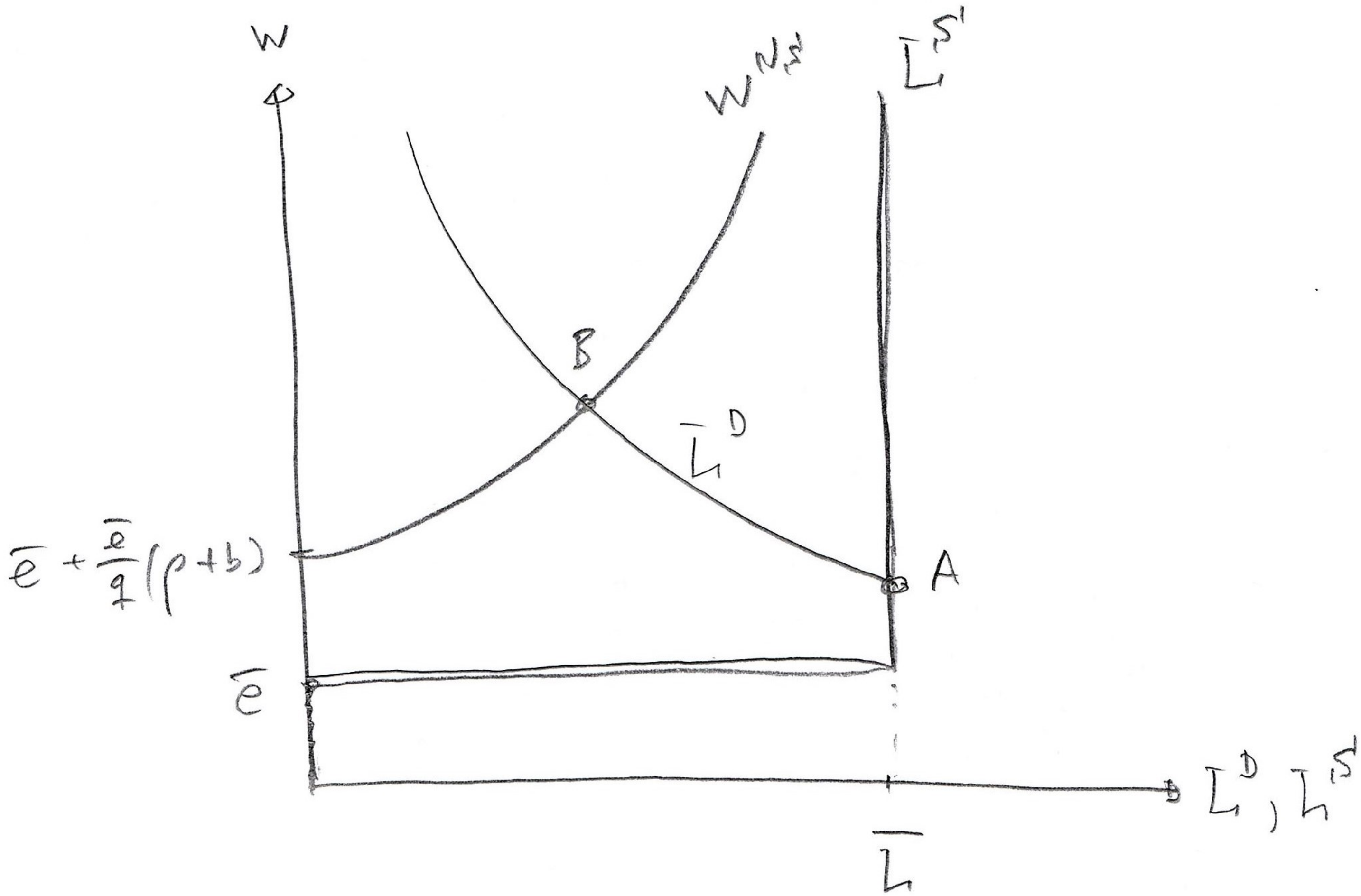

Figure 1: Illustration of equilibrium in Shapiro-Stiglitz model.