Midterm Exam – Econ 4020
29 October 2009
Department of Economics
York University

1. The Solow model [10 marks]

Consider the Solow model, where k evolves according to $\dot{k} = \phi(k)$, where

$$\phi(k) = sk^{\alpha} - (n + g + \delta)k.$$

The linearization of this equation around the steady-state level of k, denoted k^*, is given by $\dot{k} = \psi(k)$, where

$$\psi(k) = \phi(k^*) + \phi'(k^*)(k - k^*).$$

(a) Find an expression for $\psi(k)$ as the product of $(k - k^*)$ and something involving α, n, g, and δ. [5 marks]

(b) Find an expression for the "gap" at time t, given by $k(t) - k^*$. Your answer should be the product of two factors: (1) the initial gap, $k(0) - k^*$; and (2) something that involves time, t, and the parameters α, n, g, and δ. To get there you must solve the linear differential equation $\dot{k} = \psi(k)$. Hint: if $\dot{x} = ax$, then $x(t) = x(0)e^{at}$. [5 marks]

2. The Diamond model [10 marks]

Consider a version of the Diamond model where the interest rate (r) and the wage rate (w) are constant, and there is no technological progress or population growth.

Income of each agent is spent on saving (S_t), and first-period consumption (C_{1t}). The first-period budget constraint can thus be written

$$C_{1t} = w - S_t.$$

In retirement, the same agent consumes C_{2t+1}, consisting of savings from working age:

$$C_{2t+1} = S_t(1 + r).$$

The utility function is given by

$$U_t = (1 - \beta)u(C_{1t}) + \beta u(C_{2t+1}),$$

where $0 < \beta < 1$. The direct utility function is given by $u(C) = \ln(C)$.

(a) Find expressions for $u'(C)$ and $u''(C)$, and their signs. (That is, determine if they are positive or negative.) Draw the graph of $u(C) = \ln(C)$ in a diagram with C on the horizontal axis. [5 marks]

(b) Find the optimal level of C_{1t} as a function of (some or all of) r, w, and β. [5 marks]

3. Endogenous growth [10 marks]

Consider an endogenous growth model. Everything that is not explicitly a function of t below is assumed to be constant and exogenous. Output of goods, $Y(t)$, is given by

$$Y(t) = A(t)(1 - a_L)L(t),$$

where $A(t)$ the level of technology, $L(t)$ is the labor force, and a_L is the share of labor used in the R&D sector. The growth rate of the labor force is n, i.e. $\dot{L}(t) = nL(t)$. Production in the R&D sector takes this form:

$$\dot{A}(t) = [a_L L(t)]^{\gamma} [A(t)]^{\theta},$$

where $\gamma > 0$ and $\theta < 1$. The growth rate of $A(t)$ is constant on the balanced growth path, denoted by g_A^*.

(a) Find an expression for g_A^* in terms of exogenous variables. [5 marks]

(b) Let $g_{Y/L}^*$ denote the growth rate of income per worker, $Y(t)/L(t)$, on the balanced growth path. Find an expression for $g_{Y/L}^*$ in terms of exogenous variables. [5 marks]

SOLUTIONS

1 (a)

$$\begin{aligned}\psi(k) &= \phi(k^*) + \phi'(k^*)\,(k-k^*)\\ &= 0 + \left[s\alpha(k^*)^{\alpha-1} - (n+g+\delta)\right](k-k^*)\\ &= \left[s\alpha\left(\frac{n+g+\delta}{s}\right) - (n+g+\delta)\right](k-k^*)\\ &= -(1-\alpha)(n+g+\delta)(k-k^*),\end{aligned}$$

where the third equality comes from $k^* = [s/(n+g+\delta)]^{1/(1-\alpha)}$. Note that $\phi(k^*) = 0$ from the definition of k^* (i.e., it's at $k = k^*$ that $\dot{k} = 0$).

(b) Let $\chi = k - k^*$ and $\lambda = (1-\alpha)(n+g+\delta)$, so that $\dot{k} = \psi(k) = -\lambda\,(k-k^*)$. Since k^* does not depend on time, it follows that

$$\dot{\chi} = \dot{k} = \psi(k) = -\lambda(k-k^*) = -\lambda\chi,$$

which has solution $\chi(t) = \chi(0)e^{-\lambda t}$, or

$$k(t) - k^* = [k(0) - k^*]\,e^{-(1-\alpha)(n+g+\delta)t}.$$

2 (a)

$$\begin{aligned}u'(C) &= \frac{1}{C} > 0\\ u''(C) &= \frac{-1}{C^2} < 0\end{aligned}$$

The figure should look as the one attached. Note that $\ln(C) < 0$ for $C < 1$.

(b) The first-order condition gives:

$$(1-\beta)\left(\frac{1}{w-S_t}\right) = \beta\left(\frac{1}{S_t(1+r)}\right)(1+r).$$

Solving for S_t gives $S_t = \beta w$, which can be substituted into $C_{1t} = w - S_t$ to give

$$C_{1t} = (1-\beta)w.$$

3 (a)

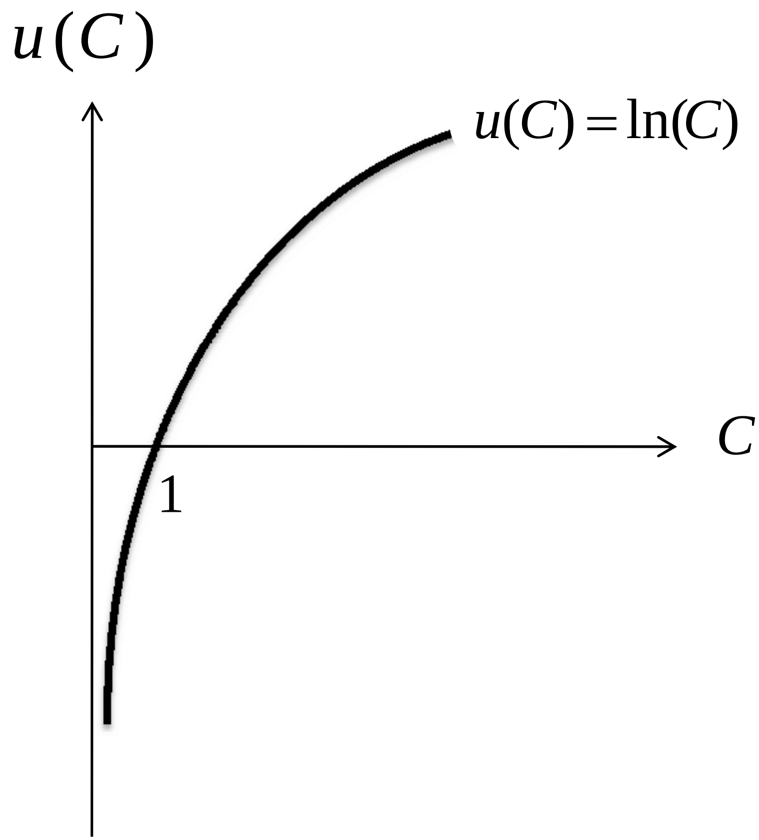
$$g_A^* = \frac{\gamma n}{1-\theta}.$$

(c) From the goods production function we get

$$\frac{Y(t)}{L(t)} = A(t),$$

which gives

$$g_{Y/L}^* = g_A^* = \frac{\gamma n}{1-\theta}.$$


$u(C)$
$u(C) = \ln(C)$
C
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Answer sheet for Problem 1, Econ 4020 Midterm, 29 October 2009

Student Name:

SID Number:

(a) Write your answer here: $\psi(k) =$

Show your calculations/motivations here:

(b) Write your answer here: $k(t) - k^* =$

Show your calculations/motivations here:

Answer sheet for Problem 2, Econ 4020 Midterm, 29 October 2009

Student Name:

SID Number:

(a) Write your answers here:

$u'(C) =$

$u''(C) =$

Draw the diagram below:

(b) Write your answer here: $C_{1t} =$

Show your calculations/motivations here:

Answer sheet for Problem 3, Econ 4020 Midterm, 29 October 2009

Student Name:

SID Number:

(a) Write your answer here: $g_A^* =$

Show your calculations/motivations here:

(c) Write your answer here: $g_{Y/L}^* =$

Show your calculations/motivations here:

Extra sheet for Problem___, Econ 4020 Midterm, 29 October 2009
Note: only one problem per sheet.

Student Name:

SID Number:

Show your calculations/motivations below: