Midterm Exam – Econ 4020
December 1, 2022
Department of Economics
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> **Instructions:** On this exam, you should *not* show how you arrived at your answer, only write the answer in the space indicated on the relevant answer sheet. Answers must be given on the correct answer sheet, one for each problem. If you make a mistake, ask for a new answer sheet. Do not fold the answer sheets or write on the back, or outside of the indicated space.

Student name: SID number:

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1. The Solow model and the CES production function [4 marks]

Let Y be output and K and L be inputs of capital and labor, respectively, and consider this CES production function:

$$Y = [\alpha K^\rho + (1-\alpha)L^\rho]^{\frac{1}{\rho}},$$

where $\alpha \in (0,1)$, $\rho < 1$, and $\rho \neq 0$.

(a) Assume that L is constant and that there is no labor augmenting productivity growth. Let K evolve over time according to $\dot{K} = sY - \delta K$, where $\delta > 0$ is the capital depreciation rate and $s \in (0,1)$ is the rate of saving. Both s and δ are constant. Under these assumptions, K will be constant in steady state. Denote that steady-state level by K^*. Find an expression for K^* in terms of (some, or all, of) L, δ, s, α, and ρ. [2 marks]

(b) Find an expression for $\lim_{\rho\to 0} Y = \lim_{\rho\to 0} [\alpha K^\rho + (1-\alpha)L^\rho]^{\frac{1}{\rho}}$. Your answer should be in terms of (some, or all, of) α, K, and L. *Hint:* If $g(0) = h(0) = 0$, then $\lim_{x\to 0} [g(x)/h(x)] = \lim_{x\to 0} [g'(x)/h'(x)]$. But you may otherwise just recall the answer from the lecture notes. [1 mark]

(c) Assume that $\rho < 0$. Find an expression for $\lim_{K\to\infty} Y = \lim_{K\to\infty} [\alpha K^\rho + (1-\alpha)L^\rho]^{\frac{1}{\rho}}$. Your answer should be in terms of (some, or all, of) ρ, α, and L. [1 mark]

2. Endogenous growth [4 marks]

Consider a Ramsey model with endogenous growth through a so-called AK technology. Output, Y, is given by

$$Y = BK, \tag{1}$$

where K is the total capital stock and B is a productivity parameter. The total labor force, denoted L, grows over time at rate $n > 0$. The capital stock per worker is given by $k = K/L$.

Let C be total consumption in the economy, so that consumption per worker becomes $c = C/L$, which evolves over time according to:

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho}{\theta},$$

where $f'(k)$ is the derivative of the intensive form production function associated with (1).

In this problem, we suppress the time arguments on Y, K, k, L, C, and c, but they all depend on time. The variables ρ, θ, B, and n are exogenous and constant.

To solve the problems below, use that $\dot{K} = Y - C$ and that C/K is constant on the balanced growth path.

(a) Find an expression for $\dot{K}/K$ on the balanced growth path. Your answer should be in terms of (some, or all, of) ρ, θ, B, and n. [2 marks]

(b) Find an expression for the ratio C/K on the balanced growth path. Your answer should be in terms of (some, or all, of) ρ, θ, B, and n. [2 marks]

3. Hazard rates [4 marks]

Consider a continuous-time model of asset pricing, like the one we used to understand the Shapiro-Stiglitz model. An asset pays different dividends depending on the state of the world, denoted A and B. The dividend is $\pi > 0$ in state A and $\pi - z$ in state B, where $z \in (0, \pi)$. The value, or price, of the asset is V_A in state A, and V_B in state B.

Let p_B be the hazard rate at which the world transitions to state B if starting in state A, and let p_A be the corresponding rate at which it transitions to state A if starting in state B.

Finally, let r be the discount rate at which future expected dividends are discounted. The variables V_A and V_B are time dependant, but we suppress the time argument, while r, p_A, p_B, π_A, and π_B are constant. We also assume that the economy is in steady state, meaning $\dot{V}_A = \dot{V}_B = 0$.

(a) Find an expression for rV_A in terms of (some, or all, of) r, p_A, p_B, π, and z, but not V_A or V_B. [1 mark]

(b) Find an expression for rV_B in terms of (some, or all, of) r, p_A, p_B, π, and z, but not V_A or V_B. [1 mark]

Hint: to solve (a)-(b), first use your intuition to write rV_A and rV_B in terms of $V_A - V_B$.

(c) Draw the graph of rV_A against z in the diagram provided. Indicate slope, vertical intercept, and what is measured on each axis. [2 marks]

Sketches of solutions

1 (Note: alternative version of the exam had λ instead of α below.)

(a)

$$K^* = sL\left(\frac{1-\alpha}{\delta^\rho - \alpha s^\rho}\right)^{\frac{1}{\rho}}$$

(b)

$$\lim_{\rho\to 0} Y = \lim_{\rho\to 0}\left[\alpha K^\rho + (1-\alpha)L^\rho\right]^{\frac{1}{\rho}} = K^\alpha L^{1-\alpha}$$

(c)

$$\lim_{K\to\infty} Y = \lim_{K\to\infty}\left[\alpha K^\rho + (1-\alpha)L^\rho\right]^{\frac{1}{\rho}} = (1-\alpha)^{\frac{1}{\rho}} L$$

2 (Note: alternative version of the exam had Z instead of B below.)

(a)

$$\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{c}}{c} + n = \frac{B-\rho}{\theta} + n$$

(b)

$$\frac{C}{K} = B - \frac{\dot{K}}{K} = B - \left(\frac{B-\rho}{\theta}\right) - n = \frac{B(\theta-1)+\rho-\theta n}{\theta}$$

3 (Note: alternative version of the exam had q instead of z below.)

(a)

$$rV_A = \pi - z\left(\frac{p_B}{r+p_A+p_B}\right)$$

(b)

$$rV_B = \pi - z + z\left(\frac{p_A}{r+p_A+p_B}\right) = \pi - z\left(\frac{r+p_B}{r+p_A+p_B}\right)$$

(c) In the exam given above, the question asked for a plot of rV_A against z, so the diagram should have rV_A on the vertical axis and z on the horizontal, and rV_A should be graphed as a straight line with slope $-p_B/(r+p_A+p_B)$, and vertical intercept π.

In the alternative version, the question asked for a plot of rV_B against q (replacing z). Then the diagram should have rV_B on the vertical axis and q on the horizontal, and rV_B should be graphed as a straight line with slope $-(r+p_B)/(r+p_A+p_B)$, and vertical intercept π.