Problems with solutions for Econ 5011

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January 6, 2005

1 Problems

1.1 Preliminaries

Problem 1 *Consider this differential equation:*

$$\overset{\bullet}{Z}(t) = a - bZ(t), \tag{1}$$

where a *and* b *are constants (i.e., independent of* t*).*
(a) Find an expression for the steady state level of $Z(t)$*; denote it* Z^**. That is,* Z^* *is the level of* $Z(t)$ *at which* $\overset{\bullet}{Z}(t) = 0$*.*
(b) Let $X(t) = Z(t) - Z^*$*. What is the steady state level of* $X(t)$*?*
(c) Find a differential equation for $X(t)$*, i.e., write* $\overset{\bullet}{X}(t)$ *as a function of* $X(t)$*.*
(d) Find a solution to the differential equation you wrote under (c). This solution should be an expression for $X(t)$ *in terms of* $X(0)$*,* t*, and things which do not depend on* t*.*
(e) Find an expression for $Z(t)$ *in terms of* $Z(0)$*,* t*, and things which do not depend on* t*.*
(f) Does $Z(t)$ *approach* Z^* *as* $t \to \infty$*?*
(g) Assume $Z(0) < Z^*$*; will* $Z(t)$ *equal* Z^* *for any finite* t*?*

Problem 2 *Let*

$$\overset{\bullet}{X}(t) = a(t)X(t),$$

where

$$a(t) = \begin{cases} \overline{a} & \text{for } t \in [0, t_0) \\ \underline{a} & \text{for } t \in [t_0, t_1] \\ \overline{a} & \text{for } t \in (t_1, \infty] \end{cases},$$

where $0 < \underline{a} < \overline{a} < \infty$*.*
(a) Show the time path of $a(t)$ *in a diagram with* t *on the horizontal axis.*
(b) Show the time path of $\ln[X(t)]$ *in a similar diagram.*
(c) Let $\overset{\bullet}{Z}(t) = \overline{a}Z(t)$*, and* $Z(0) = X(0)$*. Find an expression for the difference* $\ln[Z(t)] - \ln[X(t)]$*, for* $t > t_1$*.*
(d) Find an expression for $Z(t)/X(t)$*, for* $t > t_1$*.*

1.2 The Solow model

Problem 3 *Let steady state output per efficient worker be a function of the saving rate:* $y^*(s) = f(k^*(s))$*, where* $k^*(s)$ *is defined from*

$$sf(k^*(s)) = (n + g + \delta)k^*(s). \tag{2}$$

Let $\alpha(k) = f'(k)k/f(k)$*. Find the elasticity of* $y^*(s)$ *with respect to* s*, i.e., find*

$$\frac{\partial y^*(s)}{\partial s}\frac{s}{y^*(s)}. \tag{3}$$

Your answer should be in terms of $\alpha(k^*)$ *only.*

Problem 4 *Recall that the golden rule level of* k*, here denoted* k^g*, is given by*

$$f'(k^g) = n + g + \delta. \tag{4}$$

Let the production function be CES:

$$f(k) = [(1-\alpha) + \alpha k^\rho]^{\frac{1}{\rho}}, \tag{5}$$

where $\rho \in (-\infty, 1)$*, and* $\alpha \in (0, 1)$*. Let the golden rule level of saving be denoted* s^g*.*
(a) Is s^g *increasing or decreasing in* n*? How does your answer depend on the sign of* ρ*?*
(b) It can be seen that s^g *equals the capital share of output in the golden rule steady state. Show that in the CES case it holds that*

$$\frac{f'(k)k}{f(k)} = \alpha\left[\frac{k}{f(k)}\right]^\rho. \tag{6}$$

1.3 The Ramsey model and applications

Problem 5 ***(The transversality condition)*** *Consider the present-value budget constraint in the Ramsey model, on per-efficient-worker form and with a finite horizon, S. We can write this as*

$$\int_0^S [w(t) - c(t)]\, e^{-R(t)+(n+g)t} dt + k(0) = 0, \tag{7}$$

where the notation is as in the book and the notes. In particular, $c(t)$ and $w(t)$ are consumption and wage in per-efficient worker terms, and

$$R(t) = \int_0^t r(\tau) d\tau.$$

The budget constraint on flow-form can be written as:

$$\dot{k}(t) = w(t) + r(t)k(t) - c(t) - (n+g)k(t). \tag{8}$$

(a) Show that

$$[w(t) - c(t)]\, e^{-R(t)+(n+g)t} = \frac{\partial \left[k(t)\, e^{-R(t)+(n+g)t}\right]}{\partial t}. \tag{9}$$

(b) Use (9) to show that we can rewrite the budget constraint in (7) as:

$$k(S)e^{-R(S)+(n+g)S} = 0. \tag{10}$$

(c) Now let the horizon go to infinity. Use (10) and $k(t) = K(t)/[A(t)L(t)]$ to derive the transversality condition in the book:

$$\lim_{S \to \infty} K(S)e^{-R(S)} = 0. \tag{11}$$

Problem 6 ***(A linearized system)*** *Consider the system of differential equations derived in the Ramsey model:*

$$\begin{aligned} \dot{c}(t) &= \tfrac{c(t)}{\theta}\left[f'(k(t)) - \rho - \theta g\right] \equiv \Psi(c(t), k(t)) \\ \dot{k}(t) &= f(k(t)) - c(t) - (n+g)k(t) \equiv \Phi(c(t), k(t)). \end{aligned} \tag{12}$$

A first-order Taylor approximation of this system is given by

$$\begin{aligned} \dot{c}(t) &= \Psi_c(c^*, k^*)[c(t) - c^*] + \Psi_k(c^*, k^*)[k(t) - k^*] \\ \dot{k}(t) &= \Phi_c(c^*, k^*)[c(t) - c^*] + \Phi_k(c^*, k^*)[k(t) - k^*]. \end{aligned} \tag{13}$$

Let

$$\frac{c^*}{\theta} f''(k^*) \equiv \gamma < 0, \tag{14}$$

and recall that $\beta = \rho - n - (1-\theta)g > 0$. *Let* $\widetilde{c}(t) = c(t) - c^*$, *and* $\widetilde{k}(t) = k(t) - k^*$, *and use the vector notation*

$$z(t) = \begin{bmatrix} \widetilde{c}(t) \\ \widetilde{k}(t) \end{bmatrix}. \tag{15}$$

(a) Find expressions for $\Psi_c(c^*, k^*)$, $\Psi_k(c^*, k^*)$, $\Phi_c(c^*, k^*)$, *and* $\Phi_k(c^*, k^*)$ *in terms of* γ *and* β.

Your answer under (a) and the linearization in (13) imply that we can write

$$\dot{z}(t) = \begin{bmatrix} \dot{\widetilde{c}}(t) \\ \dot{\widetilde{k}}(t) \end{bmatrix} = \mathbf{B}z(t), \tag{16}$$

where $\mathbf{B}$ *is a* 2×2 *matrix.*
(b) Write $\mathbf{B}$ *(i.e., all its four elements) in terms of* γ *and* β.

In steady state it must hold that $\widetilde{c}(t) = \widetilde{k}(t) = 0$. *We are going to focus on linear paths leading to (or from) steady state. On these paths the ratio*

$\widetilde{c}(t)/\widetilde{k}(t)$ must be constant. This implies that $\widetilde{c}(t)$ and $\widetilde{k}(t)$ change at the same rate; call that rate μ:

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \frac{\dot{\widetilde{k}}(t)}{\widetilde{k}(t)} = \mu. \tag{17}$$

There are actually two such paths, each with a distinct rate of convergence, μ. We shall now see that these μ's are the eigenvalues of $\mathbf{B}$.
(c) Show that these μ's must satisfy $[\mathbf{B} - \mu\mathbf{I}]\, z(t) = 0$, for $z(t) \neq 0$.

Now consider any 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(d) Show that $\det[\mathbf{A} - \lambda\mathbf{I}] = 0$ *can be written as:*[1]

$$\lambda^2 - \lambda \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0 \tag{18}$$

(e) Use your insight in (d) to find both eigenvalues of $\mathbf{B}$. You may denote them μ_1 and μ_2.
(f) Which eigenvalue is associated with a stable (convergent) path, and which eigenvalue is associated with an unstable (divergent) path?

Problem 7 ***(More on eigenvalues and phase diagrams)*** *Let $\dot{x}(t) = -\alpha y(t)$, and $\dot{y}(t) = \beta x(t)$, where α and β are strictly positive constants.*
(a) Write this system on matrix form.
(b) Does the transition matrix have any real eigenvalues?
(c) Illustrate the dynamics of x and y in a phase diagram.

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[1] The terminology and notation should be familiar: $\det(\mathbf{X})$ denotes the determinant, and $\mathrm{tr}(\mathbf{X})$ the trace, of $\mathbf{X}$. That is, if

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

then $\det(\mathbf{X}) = x_{11}x_{22} - x_{12}x_{21}$, and $\mathrm{tr}(\mathbf{X}) = x_{11} + x_{22}$.
$\mathbf{I}$ is an identity matrix (with ones on the diagonal and zeros elsewhere).

Problem 8 ***(The present-value Hamiltonian)*** *Consider the following optimization problem:*

$$\begin{aligned} &\max_{c(t),k(t)} \int_0^\infty \Pi(c(t),k(t),t)dt \\ &\text{subject to } \overset{\bullet}{k}(t) = \xi(c(t),k(t),t), \end{aligned} \tag{19}$$

where $\Pi(\bullet)$ *and* $\xi(\bullet)$ *are functions;* $k(t)$ *is called the state variable, and* $c(t)$ *the control variable. The present-value Hamiltonian associated with this problem is given by*

$$H(c(t),k(t),\lambda(t),t) = \Pi(c(t),k(t),t) + \lambda(t)\xi(c(t),k(t),t), \tag{20}$$

where $\lambda(t)$ *is called the costate variable. It can be shown that the solution to this optimization problem is given by the following conditions:*

$$\begin{aligned} &H_c(c(t),k(t),\lambda(t),t) = 0 \\ &H_k(c(t),k(t),\lambda(t),t) = -\overset{\bullet}{\lambda}(t) \\ &\lim_{t\to\infty} k(t)\lambda(t) = 0 \end{aligned} \tag{21}$$

(a) Set up the Hamiltonian associated with the consumer choice problem examined in the book and in class:

$$\begin{aligned} &\max_{c(t),k(t)} \int_0^\infty e^{-\beta t}\frac{[c(t)]^{1-\theta}}{1-\theta}dt \\ &\text{subject to } \overset{\bullet}{k}(t) = w(t) + r(t)k(t) - c(t) - (n+g)k(t). \end{aligned} \tag{22}$$

(b) Apply the optimality conditions in (21) to the Hamiltonian set up under (a), and $\beta = \rho - n - (1-\theta)g$*, to derive the Euler equation:*

$$\frac{\overset{\bullet}{c}(t)}{c(t)} = \frac{1}{\theta}\left[r(t) - \rho - \theta g\right] \tag{23}$$

(c) Use your answer in (b) to show that the transversality condition [the third condition in (21)] can be written as $\lim_{t\to\infty} K(t)e^{-R(t)} = 0$*, where (recall)* $R(t) = \int_0^t r(\tau)d\tau$*. Hints: note that* $k(t) = K(t)/[A(t)L(t)]$*; that* $A(t)$ *and* $L(t)$ *grow at rates* g *and* n*, respectively; and that* $\lambda(0)$*,* $A(0)$*,* $L(0)$ *are all different from zero.*

1.4 Tobin's q

Problem 9 *(**More Hamiltonians**) Consider the optimization problem in a model with installation costs of capital:*

$$\begin{gathered}\max_{I(t),k(t)} \int_0^{\infty} e^{-rt}\left[\pi\{K(t)\}k(t) - I(t) - C(I(t))\right] dt \\ \textit{subject to} \;\; \overset{\bullet}{k}(t) = I(t).\end{gathered} \tag{24}$$

(a) Set up the present-value Hamiltonian associated with this problem, following the pattern in Problem 8. Denote the costate variable $\lambda(t)$. Which is the control variable, and which is the state variable?
(b) Find the optimality conditions corresponding to (21).
(c) Now define $q(t) = e^{rt}\lambda(t)$. Show that the optimality conditions derived under (b) can be rewritten as (8.18), (8.19), and (8.20) in the book.

1.5 Labor market models

Problem 10 *Consider the Shapiro-Stiglitz model. The value of having a job and exerting effort at some point in time t, which we denote $V^E(t)$, can be approximated as*

$$V^E(t) = (w - \overline{e})\Delta t + e^{-\rho\Delta t} E_t\{V(t + \Delta t)|E\}, \tag{25}$$

where Δt is some (small) time interval; $e^{-\rho\Delta t}$ is the relative weight put on utility at time $t + \Delta t$; and we can call $E_t[V(t + \Delta t)|E]$ the expected value at time $t + \Delta t$, conditional on "being in state E" (having a job and exerting effort) at time t.
The probability of having the job at time $t + \Delta t$ is given by $e^{-b\Delta t}$, where b is the job separation rate. We let $V^U(t + \Delta t)$ be the value of being unemployed at time $t+\Delta t$. Likewise, $V^E(t+\Delta t)$ is the corresponding value of still having the job.
(a) Find an expression for $E_t[V(t + \Delta t)|E]$ in terms of the probability $e^{-b\Delta t}$, and the value of each of the outcomes, $V^E(t + \Delta t)$ and $V^U(t + \Delta t)$.
(b) Find an expression for

$$\frac{V^E(t) - V^E(t + \Delta t)}{\Delta t} \tag{26}$$

in terms of w, $\overline{e}$, b, ρ, Δt, $V^E(t + \Delta t)$, $V^U(t + \Delta t)$.
(c) What does (26) approach as Δt goes to zero?

Your answer to (b) should look like this:

$$\begin{aligned}\frac{V^E(t+\Delta t)-V^E(t)}{\Delta t} &= w - \overline{e} - \left\{\frac{1-e^{-(\rho+b)\Delta t}}{\Delta t}\right\} V^E(t + \Delta t) \\ &+ e^{-\rho\Delta t}\left\{\frac{1-e^{-b\Delta t}}{\Delta t}\right\} V^U(t + \Delta t).\end{aligned} \tag{27}$$

(d) Letting Δt go to zero in (27), using l'Hôpital's rule[2], and your answer under (c), find an expression for $\rho V^E(t)$ which looks like in the book and the notes but now also contains a term $\dot{V}^E(t)$.

[2] l'Hôpital's rule states that, if $g(0) = h(0) = 0$, then

$$\lim_{x\to 0}\frac{g(x)}{h(x)} = \lim_{x\to 0}\frac{g'(x)}{h'(x)}.$$

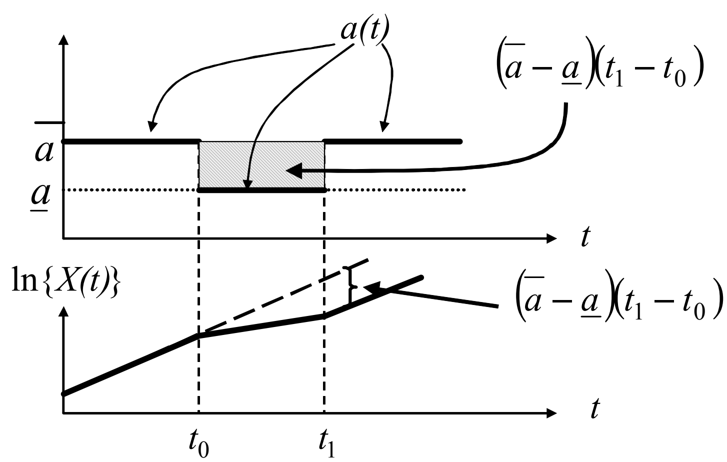
2 Solutions

Solution to Problem 1

(a) $Z^* = a/b$.
(b) $X^* = 0$.
(c) $\dot{X}(t) = -bX(t)$.
(d) $X(t) = X(0)e^{-bt}$.
(e) $Z(t) = Z^* + X(t) = a/b + X(0)e^{-bt} = \frac{a}{b} + \left[Z(0) - \frac{a}{b}\right] e^{-bt}$.
(f) Yes, since $\lim_{t\to\infty} e^{-bt} = 0$; see the answer to (e) above.
(g) No, if $Z(0) < a/b$, it holds that $Z(t) < a/b$ for all finite t, since $e^{-bt} > 0$.

Solution to Problem 2

(a)-(b) see below:



(c) $\ln[Z(t)] - \ln[X(t)] = (\overline{a} - \underline{a})(t_1 - t_0)$
(d) $Z(t)/X(t) = e^{(\overline{a}-\underline{a})(t_1-t_0)}$

Solution to Problem 3

The see textbook for details:

$$\frac{\partial y^*(s)}{\partial s}\frac{s}{y^*(s)} = \frac{\alpha(k^*)}{1-\alpha(k^*)} \tag{28}$$

Solution to Problem 4

(a) The 4010 midterm 2003 shows that

$$s^g = \left[\frac{\alpha}{(n+g+\delta)^\rho}\right]^{\frac{1}{1-\rho}}, \tag{29}$$

which tells us that s^g is increasing (decreasing) in n if $\rho < 0$ ($\rho > 0$).

(b) Note that

$$f'(k) = \frac{1}{\rho}\left[(1-\alpha) + \alpha k^\rho\right]^{\frac{1-\rho}{\rho}} \alpha\rho k^{\rho-1} = \left[f(k)\right]^{1-\rho} \alpha k^{\rho-1} = \alpha\left[\frac{k}{f(k)}\right]^{\rho-1}. \tag{30}$$

Multiplying by $k/f(k)$ gives the capital share as in (6).

Solution to Problem 5

(a) Use (8) to see that

$$\dot{k}(t) - r(t)k(t) + (n+g)k(t) = w(t) - c(t) \tag{31}$$

Differentiating $k(t)\,e^{-R(t)+(n+g)t}$ with respect to t, we get

$$\begin{aligned} &\dot{k}(t)e^{-R(t)+(n+g)t} + \left[k(t)\left\{-\dot{R}(t) + n + g\right\}e^{-R(t)+(n+g)t}\right] \\ &= \left[\dot{k}(t) - r(t)k(t) + (n+g)k(t)\right]e^{-R(t)+(n+g)t} \\ &= [w(t) - c(t)]\,e^{-R(t)+(n+g)t} \end{aligned} \tag{32}$$

where the first equality uses $\dot{R}(t) = r(t)$ and the last equality uses (31).

(b) We can write

$$\begin{aligned} 0 &= \int_0^S [w(t) - c(t)]\,e^{-R(t)+(n+g)t}dt + k(0) \\ &= \int_0^S \left[\frac{\partial\left[k(t)e^{-R(t)+(n+g)t}\right]}{\partial t}\right]dt + k(0) \\ &= \left[k(S)\,e^{-R(S)+(n+g)S}\right] - \left[k(0)\,e^{-R(0)+(n+g)0}\right] + k(0) \\ &= k(S)\,e^{-R(S)+(n+g)S} \end{aligned} \tag{33}$$

where the second equality uses (9) and the third equality uses $R(0) = 0$.

(c) Since $k(S) = K(S)/[A(S)L(S)]$, and $A(S)L(S) = A(0)L(0)e^{(n+g)S}$, we can write (33) as $K(S)e^{-R(S)} = 0$. Letting S go to infinity gives (11)

Solution to Problem 6

(a) Taking the derivatives of (12) and then imposing steady state we get:

$$\begin{aligned}\Psi_c(c^*,k^*) &= \tfrac{1}{\theta}\left[f'(k^*)-\rho-\theta g\right]=0\\ \Psi_k(c^*,k^*) &= \tfrac{c^*}{\theta}f''(k^*)=\gamma\\ \Phi_c(c^*,k^*) &= -1\\ \Phi_k(c^*,k^*) &= f'(k^*)-(n+g)=\rho+\theta g-(n+g)=\beta,\end{aligned} \tag{34}$$

where the last equality uses $f'(k^*)=\rho+\theta g$ and the definition of β.

(b) The answer under (a) gives:

$$\mathbf{B}=\begin{bmatrix} 0 & \gamma \\ -1 & \beta \end{bmatrix} \tag{35}$$

(c) From (17) we see that $\dot{\widetilde{c}}(t)=\mu\widetilde{c}(t)$ and $\dot{\widetilde{k}}(t)=\mu\widetilde{k}(t)$, so we can write:

$$\underbrace{\begin{bmatrix} \dot{\widetilde{c}}(t) \\ \dot{\widetilde{k}}(t) \end{bmatrix}}_{\dot{z}(t)}=\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}\underbrace{\begin{bmatrix} \widetilde{c}(t) \\ \widetilde{k}(t) \end{bmatrix}}_{z(t)}=\mu\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}}z(t). \tag{36}$$

We can then use (36) and (16) to write

$$\dot{z}(t)=\mu\mathbf{I}z(t)=\mathbf{B}z(t) \tag{37}$$

or

$$\left[\mathbf{B}-\mu\mathbf{I}\right]z(t)=0. \tag{38}$$

(d) Write:

$$\mathbf{A}-\mu\mathbf{I}=\begin{bmatrix} a_{11}-\mu & a_{12} \\ a_{21} & a_{22}-\mu \end{bmatrix} \tag{39}$$

so that

$$\begin{aligned}
&\det\left[\mathbf{A}-\mu\mathbf{I}\right] \\
&= (a_{11}-\mu)(a_{22}-\mu) - a_{21}a_{12} \\
&= a_{11}a_{22} - \mu a_{22} + \mu^2 - \mu a_{11} - a_{21}a_{12} \\
&= \mu^2 - \mu\underbrace{(a_{11}+a_{22})}_{\mathrm{tr}(\mathbf{A})} + \underbrace{(a_{11}a_{22}-a_{21}a_{12})}_{\det(\mathbf{A})} = 0
\end{aligned} \tag{40}$$

(e) From (35) it is seen that $\mathrm{tr}(\mathbf{B}) = \beta$, and $\det(\mathbf{B}) = \gamma$. Similar to (18), the characteristic polynomial is here given by

$$\mu^2 - \beta\mu + \gamma = 0 \tag{41}$$

which has solutions

$$\begin{aligned}
\mu_1 &= \tfrac{\beta-\sqrt{\beta^2-4\gamma}}{2} < 0 \\
\mu_2 &= \tfrac{\beta+\sqrt{\beta^2-4\gamma}}{2} > 0
\end{aligned} \tag{42}$$

where the inequalities follows from $\gamma < 0$, implying that $\sqrt{\beta^2-4\gamma} > \beta$.

(f) Note that $\dot{\widetilde{c}}(t) = \mu\widetilde{c}(t)$ has solution $\widetilde{c}(t) = \widetilde{c}(0)e^{\mu t}$. Thus $\lim_{t\to\infty}\widetilde{c}(t) = 0$ and $\lim_{t\to\infty} c(t) = c^*$ only if $\mu < 0$. So the negative eigenvalue is the stable one, and the positive eigenvalue is the unstable one.

Solution to Problem 7

(a) On matrix form it becomes

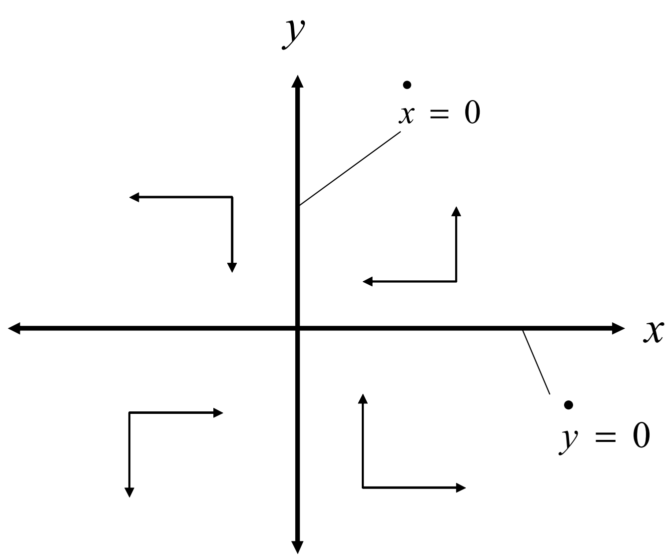
$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\alpha \\ \beta & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(b) The characteristic equation becomes

$$\lambda^2 - \text{tr}(\mathbf{A}) + \det(\mathbf{A}) = \lambda^2 + \alpha\beta = 0$$

which does not have any real solution; there are no real eigenvalues, since $\alpha\beta > 0$.

(c) The $(\dot{x} = 0)$-locus lies on the y-axis, and the $(\dot{y} = 0)$-locus lies on the x-axis. The trajectories are cyclical.



Solution to Problem 8

(a) The Hamiltonian becomes

$$H(c(t),k(t),\lambda(t),t) = \tfrac{e^{-\beta t}[c(t)]^{1-\theta}}{1-\theta}$$
$$+\lambda(t)\left[w(t)+r(t)k(t)-c(t)-(n+g)k(t)\right]. \tag{43}$$

(b) The first two optimality conditions are

$$H_c(c(t),k(t),\lambda(t),t) = e^{-\beta t}[c(t)]^{-\theta}-\lambda(t)=0$$
$$H_k(c(t),k(t),\lambda(t),t) = \lambda(t)\left[r(t)-(n+g)\right] = -\dot{\lambda}(t). \tag{44}$$

We can use (44) to write

$$e^{-\beta t}[c(t)]^{-\theta} = \lambda(t)$$

$$-\beta t - \theta\ln[c(t)] = \ln[\lambda(t)]$$

$$-\beta-\theta\tfrac{\dot{c}(t)}{c(t)} = \tfrac{\dot{\lambda}(t)}{\lambda(t)}$$

$$-\beta-\theta\tfrac{\dot{c}(t)}{c(t)} = -\left[r(t)-(n+g)\right]$$

$$-\underbrace{\left[\rho-n-(1-\theta)g\right]}_{=\beta}-\theta\tfrac{\dot{c}(t)}{c(t)} = -\left[r(t)-(n+g)\right] \tag{45}$$

$$-\left(\rho+\theta g\right)-\theta\tfrac{\dot{c}(t)}{c(t)} = -r(t)$$

$$\tfrac{\dot{c}(t)}{c(t)} = \tfrac{1}{\theta}\left[r(t)-\rho-\theta g\right]$$

(c) The second line in (44) gives a differential equation for $\lambda(t)$:

$$\dot{\lambda}(t) = \lambda(t)\left[n+g-r(t)\right] \tag{46}$$

which has solution

$$\lambda(t) = \lambda(0)e^{(n+g)t-R(t)} \tag{47}$$

We now see that

$$\begin{aligned} 0 &= \lim_{t\to\infty} k(t)\lambda(t) \\ &= \lambda(0)\lim_{t\to\infty} k(t)e^{(n+g)t-R(t)} \\ &= \underbrace{\left(\frac{\lambda(0)}{A(0)L(0)}\right)}_{\neq 0}\lim_{t\to\infty} K(t)e^{-R(t)} \end{aligned} \tag{48}$$

where the last line uses $k(t) = K(t)/[A(t)L(t)] = K(t)e^{-(n+g)t}/[A(0)L(0)]$.

Solution to Problem 9

(a) The control variable is $I(t)$ and the state variable is $k(t)$. The Hamiltonian becomes:

$$H(k(t), c(t), \lambda(t)) =$$
$$e^{-rt}\left[\pi\{K(t)\}k(t) - I(t) - C(I(t))\right] + \lambda(t)I(t) \tag{49}$$

(b) The optimality conditions become

$$H_I(\cdot) = e^{-rt}\left[-1 - C'(I(t))\right] - \lambda(t) = 0$$
$$H_k(\cdot) = e^{-rt}\pi\{K(t)\} = -\dot{\lambda}(t) \tag{50}$$
$$\lim_{t\to\infty} \lambda(t)k(t) = 0$$

(c) It is given that $q(t) = e^{rt}\lambda(t)$. The first line in (50) becomes

$$1 + C'(I(t)) = e^{rt}\lambda(t) = q(t) \tag{51}$$

which is the same as (8.18). Then the third line becomes

$$\lim_{t\to\infty} \lambda(t)k(t) = \lim_{t\to\infty} e^{-rt}q(t)k(t) = 0 \tag{52}$$

which is the same as (8.20). Then, note that

$$q(t) = e^{rt}\lambda(t)$$
$$\ln[q(t)] = \ln[e^{rt}\lambda(t)] = rt + \ln[\lambda(t)]$$
$$\frac{\dot{q}(t)}{q(t)} = r + \frac{\dot{\lambda}(t)}{\lambda(t)}$$
$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{\dot{q}(t)}{q(t)} - r \tag{53}$$
$$\dot{\lambda}(t) = \lambda(t)\left[\frac{\dot{q}(t)}{q(t)} - r\right] = q(t)e^{-rt}\left[\frac{\dot{q}(t)}{q(t)} - r\right]$$
$$e^{rt}\dot{\lambda}(t) = q(t)\left[\frac{\dot{q}(t)}{q(t)} - r\right] = \dot{q}(t) - rq(t)$$

so the second line in (50) becomes

$$\pi\{K(t)\} = -e^{rt}\dot{\lambda}(t) = rq(t) - \dot{q}(t) \tag{54}$$

which is (8.19) in the book.

Solution to Problem 10

(a) As follows:

$$E_t[V(t+\Delta t)|E] = e^{-b\Delta t}V^E(t+\Delta t) + \{1-e^{-b\Delta t}\}V^U(t+\Delta t). \tag{55}$$

(b) Use (55) and (25):

$$\begin{aligned}\tfrac{V^E(t)-V^E(t+\Delta t)}{\Delta t} &= w - \overline{e} - \left\{\tfrac{1-e^{-(\rho+b)\Delta t}}{\Delta t}\right\} V^E(t+\Delta t) \\ &+ e^{-\rho\Delta t}\left\{\tfrac{1-e^{-b\Delta t}}{\Delta t}\right\} V^U(t+\Delta t).\end{aligned} \tag{56}$$

(c) The expression is simply the definition of a derivative:

$$\begin{aligned}\lim_{\Delta t\to 0}\left\{\tfrac{V^E(t)-V^E(t+\Delta t)}{\Delta t}\right\} &= \\ -\lim_{\Delta t\to 0}\left\{\tfrac{V^E(t+\Delta t)-V^E(t)}{\Delta t}\right\} &= -\dot{V^E}(t)\end{aligned} \tag{57}$$

(d) Using l'Hôpital's rule:

$$\begin{aligned}\lim_{\Delta t\to 0}\left\{\tfrac{1-e^{-(\rho+b)\Delta t}}{\Delta t}\right\} &= \lim_{\Delta t\to 0}\left\{\tfrac{(\rho+b)e^{-(\rho+b)\Delta t}}{1}\right\} = \rho + b \\ \lim_{\Delta t\to 0}\left(e^{-\rho\Delta t}\left\{\tfrac{1-e^{-b\Delta t}}{\Delta t}\right\}\right) &= \underbrace{\left[\lim_{\Delta t\to 0} e^{-\rho\Delta t}\right]}_{=1}\underbrace{\left[\lim_{\Delta t\to 0}\left\{\frac{1-e^{-b\Delta t}}{\Delta t}\right\}\right]}_{=b} = b\end{aligned} \tag{58}$$

and (of course) $\lim_{\Delta t\to 0} V^i(t+\Delta t) = V^i(t)$, for $i = E, U$. Using (56) to (58) we get:

$$-\dot{V^E}(t) = w - \overline{e} - (\rho+b)V^E(t) + bV^U(t) \tag{59}$$

or

$$\rho V^E(t) = w - \overline{e} + \dot{V^E}(t) - b\left[V^E(t) - V^U(t)\right]. \tag{60}$$