Econ 5011 - Final Exam
23 April 2004

Part A – do all 3 problems

Problem 1. Consider the standard Ramsey model. Notation is standard, and the system of differential equations characterizing the dynamics of the economy is given by

$$\frac{\dot{c}}{c} = \frac{f'(k)-\rho-\theta g}{\theta}$$
$$\dot{k} = f(k) - c - (n+g)k$$

where it is assumed that $\rho > n + (1-\theta)g$. *Note: in the questions below, you only need to draw the diagram; you do not need to show any derivations.*
(a) Draw a phase diagram for this system, showing the saddle path leading to the steady state. [2 marks]
(b) Consider an economy situated in steady state. At some point in time, τ, population growth (n) increases. Use a diagram to show the time path for consumption per effective worker (c). The horizontal axis should show time, t. [4 marks]
(c) Do the same as in (b) but show the time path for k. [4 marks]

Problem 2. Consider a planner maximizing

$$V_0 = \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\theta}}{1-\theta} \right)$$

subject to $K_{t+1} = AK_t^{\alpha} - C_t$, where AK_t^{α} is output, C_t consumption, and K_t the capital stock. There is no leisure and population is constant and normalized to one. Everything is deterministic so you do not need to worry about the expectations operators.
(a) Set up the Bellman equation associated with this problem. This should take the form: $V(K_t) = \max\limits_{C_t} \Psi\left(C_t, V(AK_t^{\alpha} - C_t)\right)$. [5 marks]
(b) Find the Euler Equation. This should be an expression containing C_t/C_{t+1}, K_{t+1}, and exogenous variables. [5 marks]

Problem 3. Consider a central banker (a policy maker) who faces a Lucas supply curve determining the output-inflation trade-off:

$$y = \overline{y} + b(\pi - \pi^e),$$

where y is the (log of the) level of output, $\overline{y}$ is the corresponding output level when $\pi = \pi^e$ (the "flexible-price" level of output), π is inflation, and π^e is expected inflation.
The central banker controls π, and aims at maximizing a welfare function:

$$\theta y - \frac{a}{2}\left[\pi - \pi^*\right]^2, \qquad (*)$$

where θ is the weight the central banker puts on output, and π^* denotes the central banker's desired level of π, and a captures the weight the central banker puts on the inflation target.

(a) What are the equilibrium levels of output and inflation under discretion? [3 marks]
(b) What are the equilibrium levels of output and inflation under commitment? [3 marks]
(c) Assume there are many candidates to become a central banker, each with the same form of welfare function as in (*), but with different θ's, on the interval $[0, \infty)$. If society has a welfare function as that in (*), what type of central banker should it choose in order to maximize equilibrium welfare under discretion? [4 marks]

Part B – do either problem 4 or problem 5

Problem 4. Consider the Diamond model. The productivity factor, A_t, equals unity (1) in all periods. The size of the young population in period t, denoted L_t, grows at rate n, i.e., $L_{t+1} = (1+n)L_t$. Lower-case letters denote per-worker terms, e.g. $k_t = K_t/L_t$.
Agents pay tax on their working income, at a rate τ. After-tax income is spent on saving (S_t), and first-period consumption (C_{1t}). The first-period budget constraint can thus be written

$$C_{1t} = (1-\tau)w_t - S_t.$$

In retirement, the same agent consumes C_{2t+1}, consisting of savings from working age, with interest, plus a pension benefit, P_{t+1}:

$$C_{2t+1} = S_t(1 + r_{t+1}) + P_{t+1},$$

where r_{t+1} denotes the interest rate on savings held from period t to period $t+1$.
Agents are atomistic and take the tax rate and the pension benefit as given. The pension budget is pay-as-you-go funded, meaning that it must balance in all periods. This means that spending on old people in period t, $L_{t-1}P_t$, equals tax receipts from young in the same period, $L_t \tau w_t$.
The utility function is given by

$$U_t = (1-\beta)\ln C_{1t} + \beta \ln C_{2t+1},$$

where $0 < \beta < 1$.
(a) Let production be Cobb-Douglas and assume full depreciation ($\delta = 1$). As always, $K_{t+1} = S_t L_t$. Find an equation for k_{t+1} in terms of k_t and exogenous parameters. The answer should look like: $k_{t+1} = Bk_t^{\alpha}$, where B depends on exogenous parameters. [5 marks]
(b) Let τ^{**} be the tax rate at which the steady-state real interest rate equals the population growth rate ($r^* = n$). Find an expression for τ^{**} in terms of exogenous parameters. (We assume that the exogenous parameters are such that $\tau^{**} > 0$.) [10 marks]

Problem 5. Consider the Taylor model of staggered price adjustment. Individuals set prices in period t for periods t and $t+1$, and they must set the same price for both periods. The price chosen by individuals setting prices in period t is denoted χ_t. We assume certainty equivalence, meaning agents set prices at the average of what they desire (or expect to desire) in the two periods, i.e.,

$$\chi_t = \frac{1}{2}\left[p_{it}^* + E_t\left(p_{it+1}^*\right)\right]$$

where p_{it}^* is the desired price for period t, and $E_t[\cdot]$ denotes expectations as of period t. The desired prices are given by

$$p_{it}^* = \phi m_t + (1-\phi)p_t$$

where p_t is the general price level, given by the average of the prices set in the current period and the previous:

$$p_t = \frac{1}{2}\left[\chi_t + \chi_{t-1}\right].$$

Finally, money supply, m_t, follows a random walk:

$$m_t = m_{t-1} + u_t,$$

where $E_{t-1}(u_t) = 0$.
One can derive an expression for χ_t on the form:

$$\chi_t = A\left[\chi_{t-1} + E_t\left(\chi_{t+1}\right)\right] + [1-2A]\, m_t$$

where A depends on ϕ.
(a) Find the expression for A. Show how you go about. [5 marks]
(b) The model has a solution on the form: $\chi_t = \lambda\chi_{t-1} + (1-\lambda)m_t$. Find an expression for λ in terms of ϕ. (Note: There are actually two λ's that solve the system, but you should derive the one which satisfies $|\lambda| < 1$.) [10 marks]

Preliminary solutions – there may be mistakes here in which case you may send me an e-mail

Problem 1 This problems is similar to Problem 2.6 in the book (which we did in class). (a) See Figure 2.5, p. 60 in Romer's book; (b) c makes a one-time jump down at time τ and stays there; (c) nothing happens to k

Problem 2 This problem is similar to how we solved the discrete-time Ramsey model (the RBC model) in Chapter 4.
(a)

$$V(K_t) = \max_{C_t} \left\{ \frac{C_t^{1-\theta}}{1-\theta} + \beta V(AK_t^{\alpha} - C_t) \right\}$$

(b) The first-order condition gives $(C_t)^{-\theta} = \beta V'(K_{t+1})$. Denoting optimal C_t by C_t^* we can write

$$V(K_t) = \frac{(C_t^*)^{1-\theta}}{1-\theta} + \beta V(AK_t^{\alpha} - C_t^*)$$

implying that $V'(K_t) = \beta V'(K_{t+1})\alpha AK_t^{\alpha-1}$, where we note that $AK_t^{\alpha} - C_t^* = K_{t+1}$, and that C_t^* must be such that $\partial \left\{ \frac{C_t^{1-\theta}}{1-\theta} + \beta V(AK_t^{\alpha} - C_t) \right\} / \partial C_t = 0$. Using $V'(K_t) = \beta V'(K_{t+1})\alpha AK_t^{\alpha-1}$ and $(C_t)^{-\theta} = \beta V'(K_{t+1})$, we get the Euler equation: $(C_t/C_{t+1})^{-\theta} = \beta\alpha AK_{t+1}^{\alpha-1}$

Problem 3. (a) $y = \overline{y}$, $\pi = \pi^* + b\theta/a$; (b) $y = \overline{y}$, $\pi = \pi^*$
(c) This follows the discussion about delegation in the book (pp. 487-489). Let θ' be the central banker's weight on output. Note that θ' differs from society's θ, so social welfare under discretion then becomes: $\theta\overline{y} - \frac{a}{2}\left[\frac{b\theta'}{a}\right]^2$; a central banker with $\theta' = 0$ will maximize social welfare. That was the answer I was looking for. However, the central banker's own welfare will be given by $\theta'\overline{y} - \frac{a}{2}\left[\frac{b\theta'}{a}\right]^2$, and this is maximized when $\theta' = \overline{y}a/b^2$. If you answered the lattar it should give you some marks.

Problem 4. (a) The derivation is in the problem sets:

$$B = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+n)\left\{\alpha + (1-\beta)\tau(1-\alpha)\right\}}.$$

(b) In steady state $k^* = B^{\frac{1}{1-\alpha}}$, and with full depreciation $1 + r^* = \alpha\,[k^*]^{\alpha-1} = \alpha/B$, so the condition for $1 + r^* = 1 + n$ can be written

$$\frac{\alpha}{B} = \frac{(1+n)\left\{\alpha + (1-\beta)\tau(1-\alpha)\right\}}{\beta(1-\tau)(1-\alpha)} = 1 + n$$

or

$$\beta(1-\tau)(1-\alpha) = \alpha + (1-\beta)\tau(1-\alpha)$$

or

$$\tau = \beta - \frac{\alpha}{1-\alpha} \equiv \tau^{**}$$

Problem 5. (a) $A = \frac{1}{2}\left[\frac{1-\phi}{1+\phi}\right]$; (b) $\lambda = \frac{1-\sqrt{\phi}}{1+\sqrt{\phi}}$. See Romer's book (pp. 288-291) for details.