Econ 5011 - Midterm Exam
11 February 2004

Problem 1. Consider the Solow model with Cobb-Douglas production, where capital per effective worker evolves according to

$$\dot{k}(t) = s\,(k(t))^{\alpha} - (n + g + \delta)k(t).$$

The notation is completely standard; for example, $(k(t))^{\alpha} = y(t)$ is income per effective worker. To get full score on (a) to (c) below *you do not need to write any equations, or explain anything; just draw the graphs correctly.*

(a) Use a diagram with s on the horizontal axis to show how steady state income per effective worker, y^*, depends on the rate of saving, s. Draw the graph for three different cases: where α is greater than, less than, and equal to $1/2$. [3 marks]

(b) Consider an economy which is in steady state, and where the rate of saving is at its golden rule level, s^{GR}. At some point in time, $\widehat{t}$, the saving rate drops to something lower than s^{GR}, and stays there forever. Show the time path of $y(t)$. [3 marks]

(c) For the same economy as in (b), depict the time path for consumption per effective worker, $c(t)$. [4 marks]

Problem 2. Consider a Ramsey model, where population is constant ($n = 0$) and there is no technological progress ($g = 0$), and no depreciation ($\delta = 0$). Utility is given by

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(c(t))dt,$$

where ρ is the utility discount rate, $c(t)$ is per-capita consumption, and $u(c(t))$ is the instantaneous utility function, which has the standard properties: $u'(c) > 0$ and $u''(c) < 0$. Define $\sigma(c)$ as

$$\sigma(c) = -\frac{u''(c)c}{u'(c)},$$

where $\lim_{c\to\infty} \sigma(c) \equiv \sigma^* > 0$.
The production function is of "AK" type, so per-capita output, $y(t)$, is given by:

$$y(t) = f(k(t)) = Ak(t),$$

where $k(t)$ is the per-worker capital stock, and A is an exogenous constant. We assume that $A > \rho$.
With this production function there is no labor income, so the budget constraint on present value form can be written

$$k(0) = \int_{t=0}^{\infty} e^{-R(t)} c(t)dt,$$

where $R(t) = \int_{\tau=0}^{t} r(\tau)d\tau$; or on "flow" form, as:

$$\dot{k}(t) = r(t)k(t) - c(t).$$

Recall that $r(t) = f'(k(t))$ is the marginal product of capital.

(a) Find an expression for $\dot{c}(t)/c(t)$ is terms of $\sigma(c(t))$ and exogenous parameters (i.e., find the Euler equation). You may solve the utility maximization problem using either a Lagrangian, or a Hamiltonian. [5 marks]

(b) Assume that $u'''(c(t)) < 0$. Show how the growth rate of $c(t)$ evolves as the economy converges to its balanced growth path? Does it increase or decrease? [5 marks]

Problem 3. Consider an endogenous growth model with R&D. The growth of the capital stock is given by:

$$\frac{\dot{K}(t)}{K(t)} = g_K(t) = \frac{c_K\,[K(t)]^{\alpha}\,[A(t)L(t)]^{1-\alpha}}{K(t)}$$

and the growth of "ideas" (or technology) is given by:

$$\frac{\dot{A}(t)}{A(t)} = g_A(t) = \frac{c_A\,[K(t)]^{\beta}\,[L(t)]^{\gamma}\,[A(t)]^{\theta}}{A(t)}$$

where c_K and c_A are expressions involving exogenous parameters.
We assume that $L(t)$ grows at rate n, and that $\theta + \beta < 1$.
The steady state levels of $g_K(t)$ and $g_A(t)$ are denoted g_K^* and g_A^*. Also, let $\widetilde{g}_K(t) = g_K(t) - g_K^*$ and $\widetilde{g}_A(t) = g_A(t) - g_A^*$. We use the vector notation:

$$z(t) = \begin{bmatrix} \widetilde{g}_K(t) \\ \\ \widetilde{g}_A(t) \end{bmatrix} \text{ and } \dot{z}(t) = \begin{bmatrix} \dot{\widetilde{g}}_K(t) \\ \\ \dot{\widetilde{g}}_A(t) \end{bmatrix}.$$

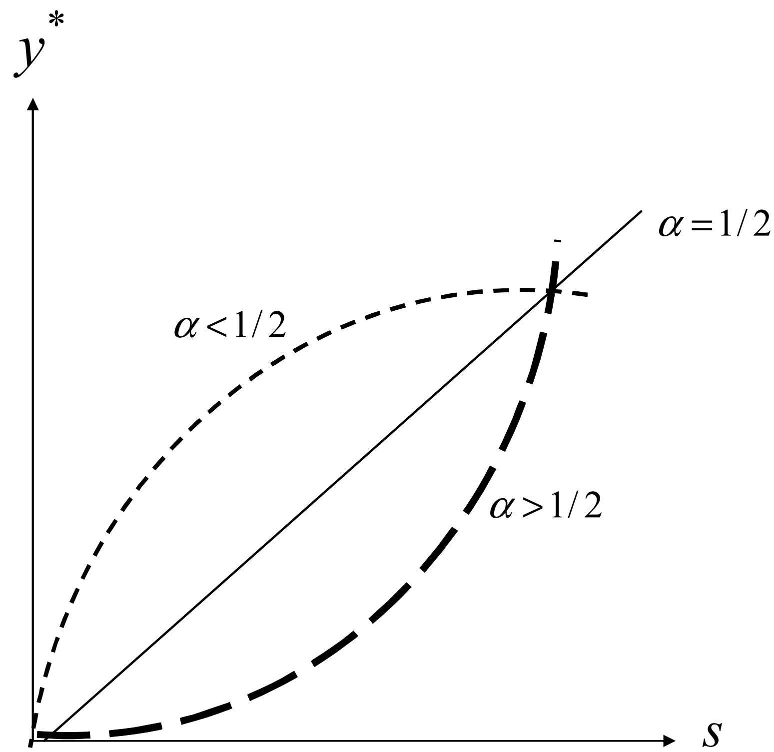
(a) Find expressions for g_K^* and g_A^* in terms of exogenous variables. [2 marks]

(b) Derive a linearized system on the form $\dot{z}(t) = Bz(t)$. The elements of the matrix B should be expressed in terms of g_K^*, g_A^*, and the exogenous variables α, β, and θ. Your answer should be such that the first row of B involves g_K^* but not g_A^*; and the second row involves g_A^* but not g_K^*; γ should not show up anywhere.[1] [4 marks]

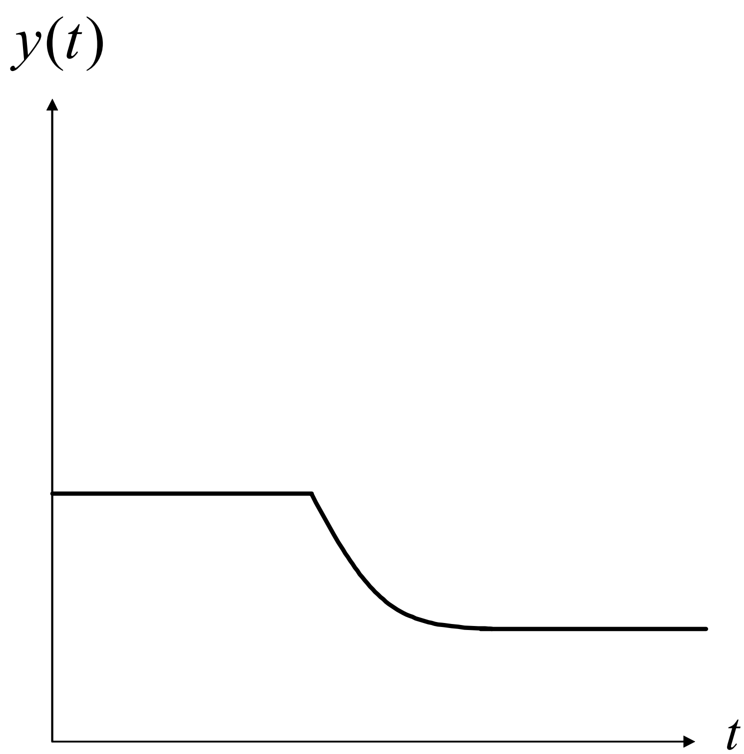
(c) Consider a path along which $\widetilde{g}_K(t)/\widetilde{g}_A(t)$ is constant over time, and let μ denote the growth rate of $\widetilde{g}_K(t)$ on this path, i.e., $\mu = \dot{\widetilde{g}}_K(t)/\widetilde{g}_K(t)$. The equation characterizing μ (the so-called characteristic polynomial) takes the form: $\mu^2 + a\mu + b = 0$. Find a and b in terms of g_K^*, g_A^*, and exogenous variables. [4 marks]

[1] That is, γ is contained in g_K^* and g_A^*.

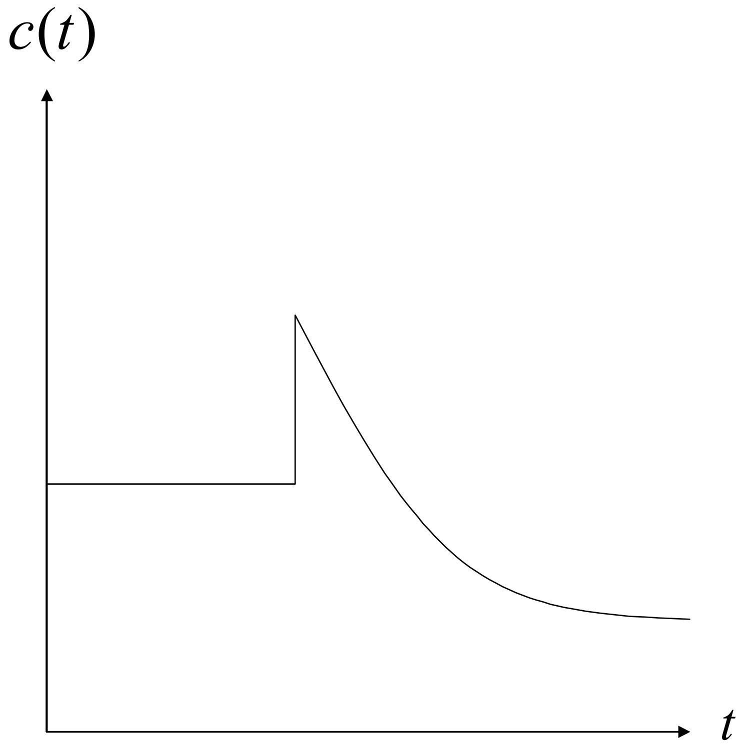
Problem 1.
(a)



(b)



(c)



Problem 2:
(a) The Lagrangian can be written:

$$\mathfrak{L} = \int_{t=0}^{\infty} e^{-\rho t} u(c(t))dt + \lambda \left(k(0) - \int_{t=0}^{\infty} e^{-R(t)} c(t)dt \right)$$

The first-order condition for $c(t)$ becomes:

$$e^{-\rho t} u'(c(t)) = \lambda e^{-R(t)}$$

Taking logarithms and differentiating with respect to t we can write:

$$-\rho + \underbrace{\frac{u''(c(t))c(t)}{u'(c(t))}}_{-\sigma(c(t))} \frac{\dot{c(t)}}{c(t)} = -R'(t) = -A$$

where the last equality uses $R'(t) = r(t) = f'(k(t)) = A$. This gives:

$$\frac{\dot{c(t)}}{c(t)} = \frac{A - \rho}{\sigma(c(t))}$$

(b) If $u'''(c(t)) < 0$ it can be seen that $\sigma'(c(t)) > 0$. Thus, since $c(t)$ is growing over time (since $A > \rho$), so is $\sigma(c(t))$. Thus, $\dot{c}(t)/c(t)$ is decreasing over time as it converges to a balanced growth path, where

$$\frac{\dot{c}(t)}{c(t)} = \frac{A-\rho}{\sigma^*}.$$

Problem 3. (a) $g_K^* = \frac{n(1-\theta+\gamma)}{1-\theta-\beta}$, $g_A^* = \frac{n(\beta+\gamma)}{1-\theta-\beta}$

(b) First note that $\dot{\tilde{g}}_K(t) = \dot{g}_K(t)$, since g_K^* is constant (and similarly for $g_A(t)$). Suppressing the time index, the linearized equation for $\dot{\tilde{g}}_K$ can be written:

$$\begin{aligned} \dot{\tilde{g}}_K &= \dot{g}_K = \left(\left.\frac{\partial \dot{g}_K}{\partial g_K}\right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) \underbrace{[g_K - g_K^*]}_{=\tilde{g}_K} \\ &+ \left(\left.\frac{\partial \dot{g}_K}{\partial g_A}\right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) \underbrace{[g_A - g_A^*]}_{=\tilde{g}_A} \end{aligned}$$

Similarly, the linearized equation for $\dot{\tilde{g}}_A$ can be written:

$$\begin{aligned} \dot{\tilde{g}}_A &= \dot{g}_A = \left(\left.\frac{\partial \dot{g}_A}{\partial g_K}\right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) \underbrace{[g_K - g_K^*]}_{=\tilde{g}_K} \\ &+ \left(\left.\frac{\partial \dot{g}_A}{\partial g_A}\right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) \underbrace{[g_A - g_A^*]}_{=\tilde{g}_A} \end{aligned}$$

Next use the equations for g_K and g_A given in the problem, to see that

$$\begin{aligned} \dot{g}_K &= g_K(1-\alpha)[g_A + n - g_K] \\ \dot{g}_A &= g_A[\beta g_K + \gamma n - (1-\theta) g_K] \end{aligned}$$

(We have derived $\dot{g}_K/g_K = (1-\alpha)[g_A + n - g_K]$ as in the book and in class, and then multiplied by g_K; likewise for g_A.)

Thus the derivatives in the expressions above can be written:

$$\left(\left.\frac{\partial \dot{g}_K}{\partial g_K}\right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) = (1-\alpha)[n + g_A^* - 2g_K^*] = -(1-\alpha)g_K^*$$

where we have used $g_A^* = g_K^* - n$, in the solution to (a) above;

$$\left(\left. \frac{\partial \overset{\bullet}{g}_K}{\partial g_A} \right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) = (1-\alpha) g_K^*$$

$$\left(\left. \frac{\partial \overset{\bullet}{g}_A}{\partial g_K} \right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) = \beta g_A^*$$

$$\begin{aligned}
\left(\left. \frac{\partial \overset{\bullet}{g}_A}{\partial g_A} \right|_{\substack{g_K = g_K^* \\ g_A = g_A^*}} \right) &= \beta g_K^* + \gamma n - 2(1-\theta) g_A^* \\
&= \beta [g_A^* + n] + \gamma n - 2(1-\theta) g_A^* \\
&= n(\beta + \gamma) + g_A^* [\beta - 2(1-\theta)] \\
&= n(\beta+\gamma) \left[1 + \frac{\beta - 2(1-\theta)}{1 - \beta - \theta} \right] \\
&= -\frac{n(\beta+\gamma)(1-\theta)}{1-\beta-\theta} = -(1-\theta) g_A^*
\end{aligned}$$

where we have used the answer in (a). The system can thus be written:

$$\underbrace{\begin{bmatrix} \overset{\bullet}{\widetilde{g}}_K \\ \overset{\bullet}{\widetilde{g}}_A \end{bmatrix}}_{\overset{\bullet}{z}} = \underbrace{\begin{bmatrix} -(1-\alpha) g_K^* & (1-\alpha) g_K^* \\ \beta g_A^* & -(1-\theta) g_A^* \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \widetilde{g}_K \\ \widetilde{g}_A \end{bmatrix}}_{z}$$

(c) **Method 1**: The μ's are simply the eigenvalues of B, and given by the solution to

$$\det[B - \mu I] = 0$$

or (as was showed in class for a general 2×2 matrix B):

$$\mu^2 - \mu \mathrm{tr}(B) + \det(B) = 0$$

or, using the expression for B above: $\mu^2 + a\mu + b = 0$, where

$$\begin{aligned}
a &= (1-\alpha) g_K^* + (1-\theta) g_A^* \\
b &= (1-\alpha) g_K^* g_A^* (1 - \beta - \theta)
\end{aligned}$$

Method 2: First note that $\widetilde{g}_K(t)/\widetilde{g}_A(t)$ being constant implies that the numerator and the denominator must grow at the same rate, μ. That is: $\overset{\bullet}{\widetilde{g}}_A(t)/\widetilde{g}_A(t) = \overset{\bullet}{\widetilde{g}}_K(t)/\widetilde{g}_K(t) = \mu$. For ease of exposition, rewrite the system of equations on the matrix form above as two separate equations:

$$\begin{aligned}\overset{\bullet}{\widetilde{g}}_K &= (1-\alpha)g_K^*\left[\widetilde{g}_A - \widetilde{g}_K\right] \\ \overset{\bullet}{\widetilde{g}}_A &= g_A^*\left[\beta\widetilde{g}_K - (1-\theta)\widetilde{g}_A\right].\end{aligned}$$

This gives:

$$\begin{aligned}\frac{\overset{\bullet}{\widetilde{g}}_K}{\widetilde{g}_K} &= (1-\alpha)g_K^*\left[\left(\frac{\widetilde{g}_A}{\widetilde{g}_K}\right) - 1\right] = \mu \\ \frac{\overset{\bullet}{\widetilde{g}}_A}{\widetilde{g}_A} &= g_A^*\left[\beta\left(\frac{\widetilde{g}_K}{\widetilde{g}_A}\right) - (1-\theta)\right] = \mu\end{aligned}$$

Using the second line:

$$\left(\frac{\widetilde{g}_K}{\widetilde{g}_A}\right) = \frac{\mu + (1-\theta)g_A^*}{\beta g_A^*}$$

which can be substituted back into the first row above:

$$\frac{\overset{\bullet}{\widetilde{g}}_K}{\widetilde{g}_K} = (1-\alpha)g_K^*\left[\left(\frac{\mu + (1-\theta)g_A^*}{\beta g_A^*}\right)^{-1} - 1\right] = \mu$$

This gives:

$$\mu^2 + \mu\underbrace{[(1-\theta)g_A^* + (1-\alpha)g_K^*]}_{=a} + \underbrace{(1-\alpha)(1-\beta-\theta)g_K^* g_A^*}_{=b} = 0$$