Problems for the graduate M.A. course in Applied Macroeconomics 5011

York University, Winter 2004

Problem Set 1 (due in class Monday January 26th, 2004):

(1) Do problems 1.2, 1.5, and 1.10 (a) in the book.

(**2**) Consider the standard Solow model. The capital stock per effective worker evolves over time according to

$$\dot{k} = sf(k) - (n + g + \delta)k.$$

Let the steady-state level of k (where $\dot{k} = 0$) be denoted k^*. The golden rule level of k is denoted k^{GR}, and given by

$$f'(k^{\mathrm{GR}}) = n + g + \delta.$$

The production function is CES:

$$f(k) = [(1 - \alpha) + \alpha k^{\rho}]^{\frac{1}{\rho}}.$$

The notation is otherwise standard.

(a) Find an expression for k^*. [We assume that $(n + g + \delta)^{\rho} > s^{\rho}\alpha$.]

(b) Find an expression for k^{GR}. [We assume that $\left(\frac{n+g+\delta}{\alpha}\right)^{\frac{\rho}{1-\rho}} > \alpha$.]

(c) Find an expression for the golden rule level of s, denoted s^{GR}.

(d) If you have made no mistakes in (a) to (c), then as ρ goes to zero, s^{GR} should go to α. Confirm this, and explain why in relation to your answer to Problem 1.5 in the book.

(3) Consider the profit maximization problem faced by an atomistic firm, taking wages, w, and the real interest rate, r, as given:

$$\max_{K,L} \pi(K, L),$$

where

$$\pi(K, L) = F(K, AL) - wAL - (r + \delta)K,$$

using the same notation as in the book and in class.

We assume that $F(K, AL)$ is a standard neo-classical production function, and thus exhibits constant returns to scale, implying that there exists some function $f(\cdot)$. such that $F(K, AL) = ALf(k)$, where $k = K/AL$.

(a) From this profit maximization problem, find an expression for w in terms of k.

(b) Find an expression for r in terms of k.

(c) Is w increasing or decreasing in k?

(d) Is r increasing or decreasing in k?

(e) What would you call w in words? *Hint:* Just the "wage rate" is not enough; something more precise is needed.

(4) Let $\dot{A}(t) = gA(t)$.

(a) Solve this differential equation. That is: find an expression for $A(t)$ in terms of $A(0)$ and t. Show each step.

(b) Let $\dot{L}(t) = nL(t)$. Find an expression for $A(t)L(t)$ in terms of $A(0)L(0)$ and t.

Problem Set 2 (due in class Monday February 9th, 2004):

(1) Do problem 8.4 in the book (on p. 407). You may also try to do the same exercise with the present value Hamiltonian.

(2) Do problem 2.6 in the book.

(3) Consider the Diamond model. We set the productivity factor A_t to unity (1) in all periods. The size of the young population in period t, denoted L_t, grows at rate n, i.e., $L_{t+1} = (1+n)L_t$. Lower-case letters denote per-worker terms, e.g. $k_t = K_t/L_t$.

Agents pay tax on their working income, at a rate τ. After-tax income is spent on saving (S_t), and first-period consumption (C_{1t}). The first-period budget constraint can thus be written

$$C_{1t} = (1 - \tau)w_t - S_t.$$

In retirement, the same agent consumes C_{2t+1}, consisting of savings from working age, with interest, plus a pension benefit, P_{t+1}:

$$C_{2t+1} = S_t(1 + r_{t+1}) + P_{t+1},$$

where r_{t+1} denotes the interest rate on savings held from period t to period $t+1$.

Agents are atomistic and take the tax rate and the pension benefit as given. The pension budget is pay-as-you-go funded, meaning that it must balance in all periods. This means that spending on old people in period t, $L_{t-1}P_t$, equals tax receipts from young in the same period, $L_t \tau w_t$.

The utility function is given by

$$U_t = (1 - \beta) \ln C_{1t} + \beta \ln C_{2t+1},$$

where $0 < \beta < 1$.

(a) Derive an expression for optimal S_t in terms of w_t, τ, P_{t+1}, r_{t+1}, and exogenous parameters.

(b) Let production be Cobb-Douglas and assume full depreciation ($\delta = 1$). As always, $K_{t+1} = S_t L_t$. Find an equation for k_{t+1} in terms of k_t and exogenous parameters. The answer should look like: $k_{t+1} = Bk_t^{\alpha}$, where B depends on exogenous parameters.

(c) How does the steady-state pension benefit, P^*, depend on population growth, n? Show how you arrived at your answer.

(4) Do problems 3.1 and 3.12 in the book.

Problem Set 3 (due in class Monday March 8th, 2004):

(1) Do problem 4.11 in the book.

(2) Do problem 6.2 in the book.

Problem Set 4 (we'll go through some of it in class but you don't have to hand in)

(1) Consider the following Fischer model of staggered price adjustment. Agents set prices for two periods ahead and the desired price for period t is given by

$$p_{it}^* = (1 - \phi)p_t + \phi m_t,$$

where p_t is the general price level and m_t is money supply. (All lower-case letters denote logarithms.) Half of the agents set prices every second period, so the general price level is given by

$$p_t = \left(\frac{1}{2}\right)\left[p_t^1 + p_t^2\right],$$

where p_t^1 is the price set in period $t-1$, and p_t^2 is the price set in period $t-2$.

Agents set their prices so that

$$p_t^1 = E_{t-1}\left[p_{it}^*\right] \text{ and } p_t^2 = E_{t-2}\left[p_{it}^*\right].$$

(a) Derive an expression for p_t^1 in terms of p_t^2 and $E_{t-1}[m_t]$. (Note that both p_t^2 and p_t^1 are known in period $t-1$.)

(b) Derive an expression for $E_{t-2}[p_t^1]$ in terms of and $E_{t-2}[m_t]$ and p_t^2.

(c) Derive an expression for p_t^2 in terms of $E_{t-2}[m_t]$.

(d) Derive an expression for y_t in terms of m_t and expectations of m_t.

(2) Do problem 3 on the Final exam from Sydney 1999

(3) Do problem 9.7 in the book

(4) Do problems 10.8 and 10.12 in the book

Solutions to parts of Problem Set 1

(Report any typos...)

(2) (a) $k^* = \left[\frac{s^\rho(1-\alpha)}{(n+g+\delta)^\rho - s^\rho\alpha}\right]^{\frac{1}{\rho}}$.

(b) $k^{\mathrm{GR}} = \left[\frac{1-\alpha}{\left(\frac{n+g+\delta}{\alpha}\right)^{\frac{\rho}{1-\rho}} - \alpha}\right]^{\frac{1}{\rho}}$.

(c) Setting $k^* = k^{\mathrm{GR}}$ gives $s^{\mathrm{GR}} = \left[\frac{\alpha}{(n+g+\delta)^\rho}\right]^{\frac{1}{1-\rho}}$, which equals α when $\rho = 0$. It can be seen that this CES production function becomes Cobb-Douglas as $\rho \to 0$.

(3) (a) $w = f(k) - f'(k)k$

(b) $r = f'(k) - \delta$

(c) $\partial w/\partial k = -f''(k)k > 0$ since $f''(k) < 0$
(d) $\partial r/\partial k = f''(k) < 0$
(f) w is the wage rate per effective worker, as opposed to Aw which is the wage rate per worker.

(4) (a) Write: $\dot{A}(t) - gA(t) = 0$. Mulitply by e^{-gt}. This gives:

$$\dot{A}(t)e^{-gt} - ge^{-gt}A(t) = \frac{\partial\left(e^{-gt}A(t)\right)}{\partial t} = 0,$$

implying that $e^{-gt}A(t)$ is constant over time; call this constant λ. That is, for all t, $e^{-gt}A(t) = \lambda$. Since this must hold also for $t = 0$, we get that $\lambda = A(0)$, i.e., $e^{-gt}A(t) = A(0)$, or

$$A(t) = A(0)e^{gt}.$$

(b) As in (a) above, we get $L(t) = L(0)e^{nt}$, and thus:

$$A(t)L(t) = A(0)e^{gt}L(0)e^{nt} = A(0)L(0)e^{(g+n)t}.$$

Solutions to Problem Set 2

(3) (a) The first-order condition generates $(1-\beta)[(1-\tau)w_t - S_t]^{-1} = \beta[S_t(1+r_{t+1}) + P_{t+1}]^{-1}(1+r_{t+1})$, which gives

$$S_t = \beta(1-\tau)w_t - (1-\beta)\left(\frac{P_{t+1}}{1+r_{t+1}}\right)$$

(b) First use $L_tP_{t+1} = L_{t+1}\tau w_{t+1}$ and $L_{t+1}/L_t = 1+n$ to derive $P_{t+1} = \tau(1-\alpha)k_{t+1}^{\alpha}(1+n)$. Then use $1+r_{t+1} = \alpha k_{t+1}^{\alpha-1}$ to see that

$$\begin{aligned} S_t &= \beta(1-\tau)w_t - (1-\beta)\frac{\tau(1-\alpha)k_{t+1}^{\alpha}(1+n)}{\alpha k_{t+1}^{\alpha-1}} \\ &= \beta(1-\tau)(1-\alpha)k_t^{\alpha} - (1-\beta)\left[\frac{\tau(1-\alpha)(1+n)}{\alpha}\right]k_{t+1} \end{aligned}$$

Then use $K_{t+1} = S_t L_t$, implying that $k_{t+1} = S_t/(1+n)$, to see that

$$k_{t+1} = \frac{\beta(1-\tau)(1-\alpha)k_t^\alpha}{1+n} - (1-\beta)\left[\frac{\tau(1-\alpha)}{\alpha}\right]k_{t+1},$$

or

$$k_{t+1}\left\{1 + (1-\beta)\left[\frac{\tau(1-\alpha)}{\alpha}\right]\right\} = \frac{\beta(1-\tau)(1-\alpha)k_t^\alpha}{1+n},$$

or

$$k_{t+1} = \frac{\frac{\beta(1-\tau)(1-\alpha)k_t^\alpha}{1+n}}{1 + (1-\beta)\left(\frac{\tau(1-\alpha)}{\alpha}\right)} = Bk_t^\alpha$$

where

$$B = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+n)\left\{\alpha + (1-\beta)\tau(1-\alpha)\right\}}.$$

(c) The steady-state pension benefit is given by

$$\begin{aligned} P^* &= \tau(1+n)w^* = \tau(1+n)(1-\alpha)\left[k^*\right]^\alpha \\ &= \tau(1+n)(1-\alpha)B^{\frac{\alpha}{1-\alpha}} = (\text{const.}) \times (1+n)^{\frac{1-2\alpha}{1-\alpha}} \end{aligned}$$

which is increasing in n if $\alpha < 1/2$, and decreasing if $\alpha > 1/2$.

Solutions to Problem Set 4

(1) (a) See Eq. (6.52) in the book.
(b) See Eq. (6.53) in the book.
(c) See Eq. (6.55) in the book.
(d) See Eq. (6.58) in the book.