Midterm Exam – Econ 5110
13 November 2012
Department of Economics
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1. Pontryagin and Ramsey [10 marks]
Consider maximizing

$$\int_0^T f(t, u(t), x(t))dt, \tag{1}$$

subject to

$$\dot{x}(t) = g(t, u(t), x(t)). \tag{2}$$

In class we saw that the Hamiltonian associated with this maximization problem could be written $H(x, u, \lambda) = f(t, u(t), x(t)) + \lambda g(t, u(t), x(t))$.
Now consider the Hamiltonian associated with (a particular version of) the continuous-time Ramsey model discussed in class:

$$H(k, c, \lambda) = e^{-\rho t}\ln(c) + \lambda[w + rk - c], \tag{3}$$

where the notation is the same as in class and the time arguments are suppressed on w, r, k, c, and λ.
(a) Write the original maximization problem in this version of the Ramsey model, according to the template in (1) and (2). (Hint: your answer should read: "maximize..., subject to...".) [3 marks]
(b) The optimality conditions associated with the Hamiltonian in (3) are

$$\begin{aligned} H_c(k, c, \lambda) &= 0, \\ H_k(k, c, \lambda) &= -\dot{\lambda}. \end{aligned}$$

Use these optimality conditions to derive an Euler equation (an expression for $\dot{c}/c$ in terms of r and ρ). [3 marks]
(c) Recall that, if we assume zero depreciation, then $r = f'(k)$, and the Euler equation implies that we can write $\dot{c} = \Psi(c, k)$. Let c^* and k^* be the steady state levels of c and k. Show that $\Psi_c(c^*, k^*) = 0$. (The sub-index indicates the variable with respect to which Ψ is differentiated.) [3 marks]
(d) We call k a state variable. What do we call c? [1 mark]

2. The Hansen-Prescott model [10 marks]

Consider the Hansen-Prescott model. Production in the Malthus and Solow sectors are given by

$$Y_{M,t} = A_{M,t} K_{M,t}^{\phi} N_{M,t}^{\mu} L^{1-\mu-\phi}$$

$$Y_{S,t} = A_{S,t} K_{S,t}^{1-\mu} N_{S,t}^{\mu}$$

where $A_{i,t}$ is total factor productivity, $K_{i,t}$ is capital, and $N_{i,t}$ is labor, in sectors $i = M, S$ (Malthus and Solow). L is the amount of land. K_t and N_t are the total amounts of capital and labor. Productivity growth is given by

$$A_{M,t+1} = \gamma_M A_{M,t},$$
$$A_{S,t+1} = \gamma_S A_{S,t},$$

where $\gamma_S > \gamma_M$.

(a) Let $z_{N,t}$ be the fraction labor in Solow sector, and $z_{K,t}$ the fraction capital in Solow sector. Thus, $K_{S,t} = z_{K,t} K_t$, etc. Write two equations which determine these two variables, $z_{N,t}$ and $z_{K,t}$, in an equilibrium where both sectors are active. You do not need to solve for $z_{N,t}$ and $z_{K,t}$, just write the two equations. The two equations should not involve N_t. [4 marks]

(b) State in words what the equations written in your answer to (a) mean. [2 marks]

(c) On a Malthusian balanced growth path all K_t and N_t are allocated to the Malthus sector. Let fertility (and thus the population growth rate) on the Malthusian balanced growth path be n^*, and the associated wage (given by the marginal product to labor in the Malthus sector) be w^*. Find an expression for n^* in terms of γ_M, μ, and ϕ. (Hint: the capital-labor ratio is constant on the Malthusian balanced growth path.) [4 marks]

3. The Lucas fertility model with property rights [10 marks]

Consider a dynastic overlapping-generations model where in period t an agent's utility, u_t, is given by

$$u_t = (1-\beta)\ln(c_t) + \beta[\gamma \ln(n_t) + u_{t+1}],$$

where $\gamma > 1$, $0 < \beta < 1$, c_t is consumption, and n_t is fertility. The budget constraint is given by

$$y_t = A x_t^{\alpha} = c_t + k n_t,$$

where k is a goods cost per child, $y_t = A x_t^{\alpha}$ is the agent's output (where $0 < \alpha < 1$), and $x_t = L/N_t$ is land per agent in the dynasty. That is, there are property rights to land, and L is the total amount of land owned by the dynasty, which is in fixed supply. Also, N_t is the dynasty's total population in period t. Since n_t is fertility and this is a one-sex model, $N_{t+1} = n_t N_t$. The Bellman equation is given by:

$$W(x_t) = \max_{n_t} \{(1-\beta)\ln(A x_t^{\alpha} - k n_t) + \beta\gamma \ln(n_t) + \beta W(x_{t+1})\}.$$

(a) Show that $x_{t+1} = x_t / n_t$. [2 marks]

(b) Find steady state income per agent, $y = A x^{\alpha}$, in terms of exogenous variables. Hint: in steady state it holds that population, N_t, is constant. [8 marks]

4. The Nunn-Qian potato paper [5 marks]
(a) The introduction of the potato outside the Americas had different effects in different parts of the world where it had previously never been grown. Why? [2.5 marks]
(b) Which two outcome variables (i.e., dependent variables) do Nunn and Qian consider when examining whether the potato did have an effect? [2.5 marks]

5. The Ashraf-Galor paper [5 marks]
(a) What source do Ashraf and Galor use for per-capita income data? (Just the name is enough.) What years do they consider? [1 mark]
(b) What does the Malthusian model predict about (steady-state) *per-capita incomes* in more technologically advanced regions or countries compared to less advanced? Is this prediction of the Malthusian model consistent with Ashraf and Galor's findings? [2 marks]
(c) What does the Malthusian model predict about (steady-state) *population densities* in more technologically advanced regions or countries compared to less advanced? Is this prediction of the Malthusian model consistent with Ashraf and Galor's findings? [2 marks]

Answer sheet for Problem____
Econ 5110, Midterm 13 November 2012

Student Name:

SID Number:

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Sketches of solutions to some problems:

1.
(a) Maximize

$$\int_0^T e^{-\rho t} \ln(c(t))dt,$$

subject to

$$\dot{k}(t) = w(t) + r(t)k(t) - c(t).$$

(b) See notes for general (non-log utility) case:
http://www.nippelagerlof.com/teaching/5110/PontryaginNotesOptimized.pdf
(c) Must write $\Psi(c,k) = c\,[f(k) - \rho]$ correctly and show that

$$\Psi_c(c^*, k^*) = [f(k^*) - \rho] = 0.$$

(d) Control variable.

2.
(a) Substitute $K_{S,t} = z_{K,t}K_t$, $K_{M,t} = (1 - z_{K,t})K_t$, $N_{S,t} = z_{N,t}N_t$, $N_{M,t} = (1 - z_{N,t})N_t$ into production functions and equalize marginal products of labor across sectors. This gives:

$$\mu A_{M,t}(1 - z_{K,t})^{\phi}(1 - z_{N,t})^{\mu-1}K_t^{\phi}N_t^{\mu-1}L^{1-\mu-\phi} = \mu A_{S,t}(z_{K,t})^{1-\mu}(z_{N,t})^{\mu-1}K_t^{1-\mu}N_t^{\mu-1},$$

or

$$A_{M,t}(1 - z_{K,t})^{\phi}(1 - z_{N,t})^{\mu-1}K_t^{\phi}L^{1-\mu-\phi} = A_{S,t}(z_{K,t})^{1-\mu}(z_{N,t})^{\mu-1}K_t^{1-\mu}, \tag{4}$$

which contains no N_t. Then equalize marginal products of capital across sectors, which gives:

$$\phi A_{M,t}(1 - z_{K,t})^{\phi-1}(1 - z_{N,t})^{\mu}K_t^{\phi-1}N_t^{\mu}L^{1-\mu-\phi} = (1-\mu)A_{S,t}(z_{K,t})^{-\mu}(z_{N,t})^{\mu}K_t^{-\mu}N_t^{\mu},$$

or

$$\phi A_{M,t}(1 - z_{K,t})^{\phi-1}(1 - z_{N,t})^{\mu}K_t^{\phi-1}L^{1-\mu-\phi} = (1-\mu)A_{S,t}(z_{K,t})^{-\mu}(z_{N,t})^{\mu}K_t^{-\mu}, \tag{5}$$

which contains no N_t. Since (4) and (5) can be used to solve for $z_{K,t}$ and $z_{N,t}$ they constitute a correct answer.
(b) Marginal products of labor and capital are equalized across sectors.
(c) First show that $N_{t+1} = n^*N_t$, and K_t/N_t being constant, together imply $K_{t+1} = n^*K_t$. Then note that $z_{K,t} = z_{N,t} = 0$ on the Malthusian BGP. Because the wage rate is constant and equal to the marginal product of labor in all periods it follows that

$$\begin{aligned} w^* &= \mu A_{M,t}K_t^{\phi}N_t^{\mu-1}L^{1-\mu-\phi} \\ &= \mu A_{M,t+1}K_{t+1}^{\phi}N_{t+1}^{\mu-1}L^{1-\mu-\phi} \\ &= \mu\gamma_M A_{M,t}(n^*K_t)^{\phi}(n^*N_t)^{\mu-1}L^{1-\mu-\phi} \\ &= \gamma_M(n^*)^{\phi+\mu-1}w^*. \end{aligned}$$

where the last equality uses the first. This generates $\gamma_M(n^*)^{\phi+\mu-1} = 1$, or

$$n^* = \gamma_M^{\frac{1}{1-\phi-\mu}}.$$

3. (a) Straightforward.
(b) The first-order condition for n_t gives

$$(1-\beta)\frac{k}{c_t} = \beta\gamma\frac{1}{n_t} + \beta W'(\frac{x_t}{n_t})\frac{-x_t}{n_t^2}.$$

The envelope theorem says that

$$W'(x_t) = (1-\beta)\frac{A\alpha x_t^{\alpha-1}}{c_t} + \beta W'(\frac{x_t}{n_t})\left(\frac{1}{n_t}\right) + 0.$$

In steady state where x is constant and $n = 1$ we get

$$(1-\beta)W'(x) = (1-\beta)\tfrac{A\alpha x^{\alpha-1}}{c},$$

$$W'(x) = \tfrac{A\alpha x^{\alpha-1}}{c}.$$

Substituting back into the first-order condition we get

$$(1-\beta)\frac{k}{c} = \beta\gamma - \beta\left[\frac{A\alpha x^{\alpha-1}}{c}\right]x,$$

or

$$\begin{aligned}(1-\beta)k &= \beta\gamma c - \beta A\alpha x^{\alpha} \\ &= \beta\gamma[y-k] - \beta\alpha y \\ &= y\beta(\gamma-\alpha) - \beta\gamma k.\end{aligned}$$

Solving for y gives:

$$y = \frac{k[1-\beta+\beta\gamma]}{\beta(\gamma-\alpha)}.$$

4. (a) Because of geographical (cross-country) variation in the suitability of local environmental conditions (primarily climate) for growing potatoes.
(b) Total country population and urbanization (i.e., people living in cities divided by total country population).

5. (a) The source is Angus Maddison. The years are 1, 1000, and 1500 CE.
(b) The Malthusian model predicts that steady-state per-capita incomes do not depend on the level of technology. Ashraf and Galor find that this is consistent with the data.
(c) The Malthusian model predicts that steady-state population density is increasing with the level of technology. Ashraf and Galor find that this is also consistent with the data.