The Hansen-Prescott (2002) model

Long-run growth model

Two sectors:

- Malthus sector using land, labor, capital
- Solow sector using labor, capital

Population evolves according to an exogenously given function of consumption (here actually wages)

Transition

- from Malthusian economy, with stagnant living standards and only Malthus sector active

- to Solow economy, growing living standards and both sectors active

Production:

$$Y_{M,t} = A_{M,t} K_{M,t}^{\phi} N_{M,t,}^{\mu} L^{1-\mu-\phi}$$

$$Y_{S,t} = A_{S,t} K_{S,t}^{1-\mu} N_{S,t}^{\mu}$$

$A_{i,t} =$ total factor productivity in sector i

$K_{i,t} =$ capital

$N_{i,t} =$ labor

$L =$ land (exogenous)

$i = M, S$ (Malthus, Solow)

Productivity growth:

$$A_{M,t+1} = \gamma_M A_{M,t},$$
$$A_{S,t+1} = \gamma_S A_{S,t},$$

$$\gamma_S > \gamma_M$$

$z_{N,t} =$ fraction labor in Solow sector

$z_{K,t} =$ fraction capital in Solow sector

Clearing factor markets:

$$z_{K,t} = 1 - V_t,$$

$$z_{N,t} = \frac{\phi(1 - V_t)}{\phi + (1 - \mu - \phi) V_t},$$

$$V_t = \min\left\{1, \left[\left(\frac{\phi}{1-\mu}\right)^{1-\mu} \frac{A_{M,t}}{A_{S,t}}\right]^{\frac{1}{1-\mu-\phi}} \frac{L}{K_t}\right\}$$

Malthus to Solow transition when low enough TFP ratio, and/or land-capital ratio

Agents

Two-period OLG

Different ways to model ownership of land

- No property rights: land controlled and shared in equal parcels by the young in any given period; left to next generation when turning old (easier approach)

- With property rights: markets for land where old agents sell to young at endogenously determined price (more difficult, done by Hansen and Prescott)

Here: no property rights

Budget constraints

$$c_{1,t} = w_t + r_{L,t} l_t - s_t$$

$$c_{2,t+1} = (1 + r_{K,t+1}) s_t$$

$s_t =$ saving

$w_t =$wage

$r_{K,t+1} =$real interest rate

$r_{L,t} =$rental price of land

$l_t = L/N_t =$ land per young agent

Utility

$$U_t = (1 - \beta) \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

Saving:

$$s_t = \beta \left[w_t + r_{L,t} l_t \right]$$

Fertility, n_t, function of wage, w_t

$$n_t = n(w_t) = \begin{cases} b + \frac{(n^* - b) w_t}{w^*} & \text{if } w_t < \eta w^* \\ d - \left(\frac{\overline{n} - 1}{\nu - \eta}\right) \frac{w_t}{w^*} & \text{if } w_t \in [\eta w^*, \nu w^*] \\ 1 & \text{if } w_t > \nu w^* \end{cases}$$

$$b = \frac{\eta n^* - \overline{n}}{\eta - 1}$$

$$d = \frac{\nu \overline{n} - \eta}{\nu - \eta}$$

Parameters w^*, $\overline{n}$, n^*, ν and η define fertility behavior

$w^* > 0$, $\overline{n} > n^*$, $\nu > \eta > 1$

Note:

- $n'(w) > 0$ for $w_t < \eta w^*$
- w^* and n^* = wage and fertility rates on a Malthusian balanced growth path (when Solow sector not active)
- Fertility peaks at $\overline{n}$, when $w_t = \eta w^*$
- Then declines until the wage rate reaches νw^*
- After that $n_t = 1$, population remains constant

Population evolves according to $N_{t+1} = N_t n_t$

Malthusian balanced growth path

Set Malthusian fertility rate, n^*, so that wages are constant (at w^*) on balanced growth path; gives

$$n^* = \gamma_M^{\frac{1}{1-\phi-\mu}}$$

Why? Use

$$Y_{Mt} = A_{M,t} K_{M,t}^{\phi} N_{M,t}^{\mu} L^{1-\mu-\phi} = A_{M,t} K_t^{\phi} N_t^{\mu} L^{1-\mu-\phi}$$

$$A_{M,t+1} = \gamma_M A_{M,t}$$

$$N_{t+1} = N_t n^*$$

$$w^* = \mu A_{M,t} K_t^{\phi} N_t^{\mu-1} L^{1-\mu-\phi} = \mu A_{M,t+1} K_{t+1}^{\phi} N_{t+1}^{\mu-1} L^{1-\mu-\phi}$$

Constant capital-labor ratio implies K_t and N_t grow at same rate, n^*

Dynamics

4 state variables that determine everything else: K_t, N_t, $A_{S,t}$, $A_{M,t}$

Last two evolve exogenously

Find dynamics of capital and population

Capital

Capital made up of previous period's saving

$$\begin{aligned} K_{t+1} &= s_t N_t \\ &= \beta \left[w_t + r_{L,t} l_t \right] N_t \\ &= \beta \left[w_t N_t + r_{L,t} L \right] \end{aligned}$$

where w_t and $r_{L,t}$ are given by the marginal products to labor and land:

$$w_t = \frac{\mu Y_{M,t}}{\left(1 - z_{N,t}\right) N_t}$$

$$r_{L,t} = \frac{(1 - \mu - \phi) Y_{M,t}}{L}$$

$$K_{t+1} = \beta \left(1 - \mu - \phi + \frac{\mu}{1 - z_{N,t}}\right) Y_{M,t}$$

Note spurt in capital accumulation when $z_{N,t}$ increases

Population

Recall exogenous fertility function, $n(w_t)$

$$N_{t+1} = N_t n(w_t)$$

where w_t is given above

Transition

The Solow sector becomes active when V_t falls below 1, which occurs when

$$A_{S,t} \geq \left(\frac{\phi}{1-\mu}\right)^{1-\mu} \left[A_{M,t}\left(\frac{L}{K_t}\right)^{1-\mu-\phi}\right].$$

Paths very easy (in principle) to simulate

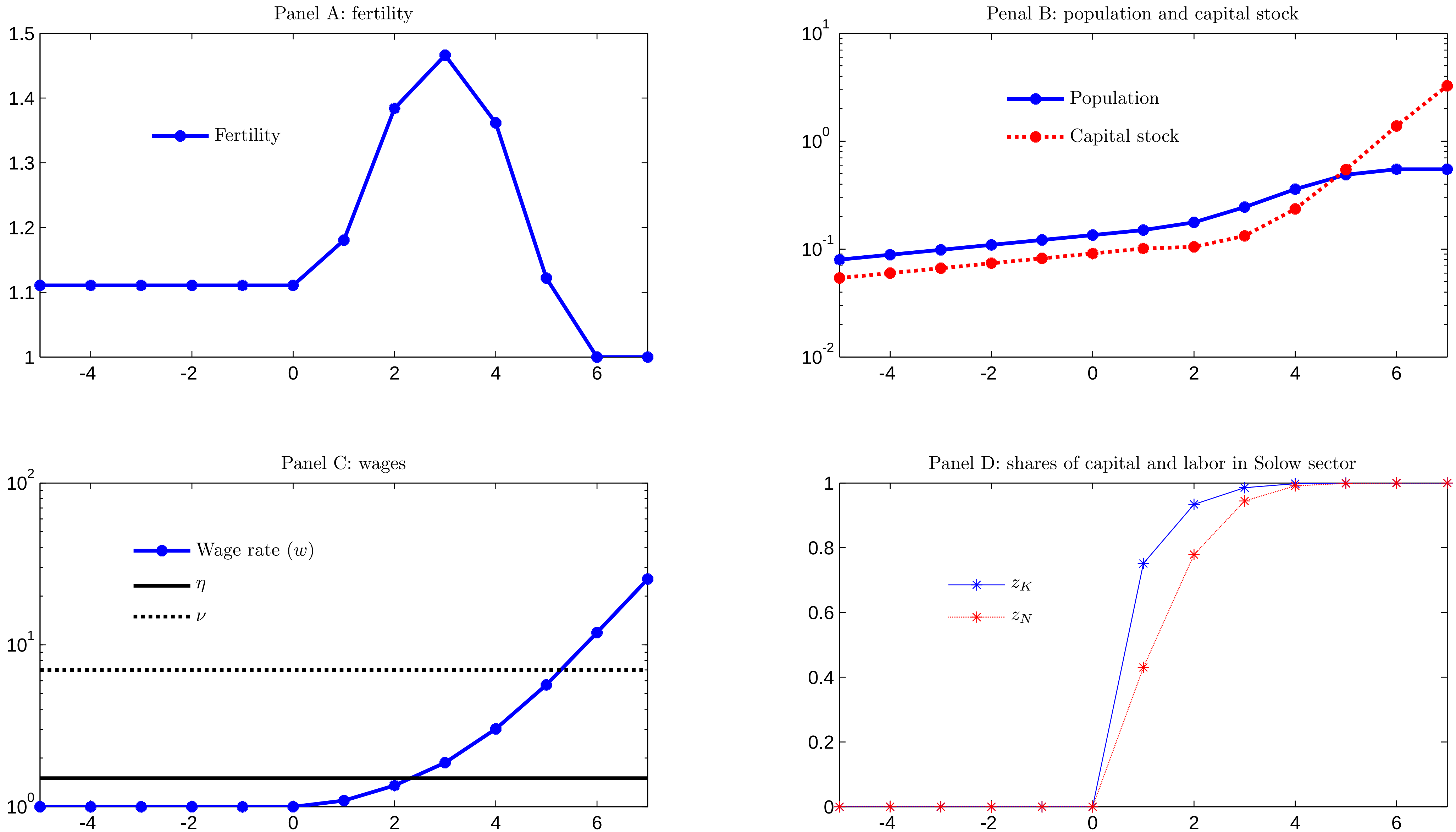
- Start with K_0, N_0, $A_{S,0} = 1$, $A_{M,0} = 1$ (pick K_0, N_0 so that Solow sector is not active and economy is on Malthusian BGP)

- Compute $z_{N,t}$, $z_{K,t}$, w_t, $r_{L,t}$, $Y_{M,t}$ for $t = 0$

- Use dynamic equations to update K_t, N_t, $A_{S,t}$, $A_{M,t}$ to $t = 1$

- Repeat

See figures for example


Panel A: fertility
Fertility
1.5
1.4
1.3
1.2
1.1
1
-4
-2
0
2
4
6
Penal B: population and capital stock
Population
Capital stock
10^{1}
10^{0}
10^{-1}
10^{-2}
Panel C: wages
Wage rate (w)
η
ν
10^{2}
10^{1}
10^{0}
Panel D: shares of capital and labor in Solow sector
z_K
z_N
1
0.8
0.6
0.4
0.2
0