Linearizing the dynamical system

Suppress "t"

(pp. 389–390 in Acemoglu's book)

$$\dot{c} = \frac{c}{\theta}\left[f'(k) - \rho - \theta g\right] \equiv \Psi(c,k)$$

$$\dot{k} = f(k) - c - (n+g)k \equiv \Phi(c,k)$$

1st-order Taylor approx. around c^*, k^*

$$\dot{c} = \Psi_c(c^*,k^*)[c - c^*] + \Psi_k(c^*,k^*)\,[k - k^*]$$

$$\dot{k} = \Phi_c(c^*,k^*)[c - c^*] + \Phi_k(c^*,k^*)\,[k - k^*]$$

Define:

$$\tilde{c} = c - c^*$$

$$\tilde{k} = k - k^*$$

Note: $\dot{\tilde{c}} = \dot{c}$, $\dot{\tilde{k}} = \dot{k}$

$$\Psi_c(c^*, k^*) = \frac{1}{\theta}\left[f'(k^*) - \rho - \theta g\right] = 0$$

$$\Psi_k(c^*, k^*) = \frac{c^*}{\theta} f''(k^*) \equiv \alpha < 0$$

$$\Phi_c(c^*, k^*) = -1$$

$$\Phi_k(c^*, k^*) = f'(k^*) - (n+g) = \rho + \theta g - (n+g) = \beta$$

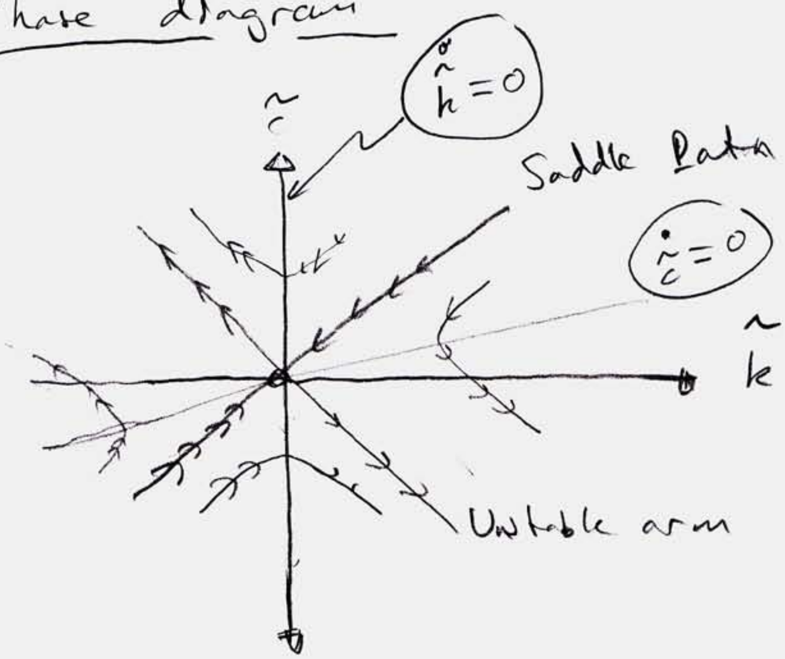
$$\dot{\tilde{c}} = \alpha \tilde{c} + \gamma \tilde{k}$$

$$\dot{\tilde{k}} = -\tilde{c} + \beta \tilde{k}$$

On Matrix form

$$\begin{bmatrix} \dot{\tilde{c}} \\ \dot{\tilde{k}} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -1 & \beta \end{bmatrix} \begin{bmatrix} \tilde{c} \\ \tilde{k} \end{bmatrix}$$

Phase diagram



$\dot{\tilde{c}} = 0 \Rightarrow \tilde{k} = 0$

$\dot{\tilde{k}} = 0 \Rightarrow \tilde{c} = \beta \tilde{k}$

Consider paths on

saddle path, or unstable arm

$$\Rightarrow \quad \frac{\tilde{c}}{\tilde{k}} = \text{const.}$$

$$\Rightarrow \quad \frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{\tilde{k}}}{\tilde{k}} = \mu \quad (\text{say})$$

$$\dot{\tilde{c}} = \mu \tilde{c} = 0 + \alpha \tilde{k}$$

$$\dot{\tilde{k}} = \mu \tilde{k} = -\tilde{c} + \beta \tilde{k}$$

<u>Divide</u> by <u>$\tilde{k}$</u>

$$\mu \left(\frac{\tilde{c}}{\tilde{k}}\right) = \alpha \quad \Rightarrow \quad \frac{\tilde{c}}{\tilde{k}} = \frac{\alpha}{\mu}$$

$$\mu = -\frac{\tilde{c}}{\tilde{k}} + \beta = -\frac{\alpha}{\mu} + \beta$$

$$\mu = \beta - \frac{\alpha}{\mu}$$

$$\mu^2 - \beta\mu + \alpha = 0$$

$$\boxed{\mu_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha}}{2}}$$

$$\alpha = \frac{f''(k^*)\, c^*}{\theta} < 0$$

$$\beta = n - \rho - (1-\theta) g > 0$$

One root > 0 ; other < 0

Using Matrix algebra

$$\underbrace{\begin{bmatrix} \dot{\hat{c}} \\ \dot{\hat{k}} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} 0 & \alpha \\ -1 & \beta \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \hat{c} \\ \hat{k} \end{bmatrix}}_{z}$$

$$\boxed{\dot{z} = B\, z}$$

Look at paths where $\dfrac{\hat{c}}{\hat{k}}$ constant

$$\left.\begin{aligned} \dot{\hat{c}} &= \mu\, \hat{c} \\ \dot{\hat{k}} &= \mu\, \hat{k} \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} \dot{\hat{c}} \\ \dot{\hat{k}} \end{bmatrix}}_{\dot{z}} = \mu \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} \hat{c} \\ \hat{k} \end{bmatrix}}_{z}$$

$$\boxed{\dot{z} = \mu I z}$$

$$\boxed{\left[B - \mu I\right] z = 0}$$

Must hold for some $z \neq 0$

μ is the <u>eigenvalue</u> of B

z is the <u>eigenvector</u> of B

To find μ, set $\mathrm{Det}(B - \mu I) = 0$

To find z, set $\left[B - \mu I\right] z = 0$