The Lucas (1988) model(s)

* Marshall Lectures at Cambridge U 1985, not written like standard article

* First part: standard (cont. time) Ramsey Model

* Issue: changes in ρ (discount rate) have no effect on sustained growth rate

* Set up model with human capital + pos. externality from human capital in production

Preliminaries

$$U = \int_0^\infty e^{-\rho t} u(c)\, dt$$

s.t. $\dot{k} = f(k) - c$

Present-value Hamiltonian

$$H = e^{-\rho t} u(c) + \lambda [f(k) - c]$$

opt. conditions

$H_c = 0$, $H_k = -\dot{\lambda}$

Current-value Hamiltonian

$$H = u(c) + \theta [f(k) - c]$$

opt. conditions

$H_c = 0$

$H_k = \rho \theta - \dot{\theta}$

Utility:

$$\boxed{U = \int_0^\infty e^{-\rho t} N \ln(c)\, dt}$$

c = consumption / worker

N = # of workers

$\frac{\dot{N}}{N} = n > \rho$ (n is λ in Lucas)

Constraints

$$\boxed{\dot{K} = K^{\alpha}[uNh]^{1-\alpha} h_a^{\gamma} - Nc}$$

(α is β in Lucas)

h_a = average level of h

γ measures size of externality

$$\boxed{\dot{h} = h\delta[1-u]}$$

Current-value Hamiltonian

$$H = N\ln(c) + \theta_1\left[\underline{K}^{\alpha}[uNh]^{1-\alpha}h_a^{\gamma} - Nc\right]$$

$$+ \theta_2\,\delta h(1-u)$$

Two types of BGP

(I) Equilibrum path:

* treat h_a as given, solve max problem
* impose $h = h_a$

(II) Optimal path

* impose $h = h_a$
* solve max problem

(I) Equil. Path

Optimality conditions

$$H_c = N\left(\frac{1}{c}\right) - N\theta_1 = 0$$

$$H_u = \theta_1 (1-\alpha) \underbrace{\underline{K}^{\alpha} (uNh)^{1-\alpha} h_a^{\gamma}}_{\underline{Y}} \left[\frac{1}{u}\right] - \theta_2 \delta h = 0$$

$$H_{\underline{K}} = \theta_1 \alpha \underbrace{\underline{K}^{\alpha} (uNh)^{1-\alpha} h_a^{\gamma}}_{\underline{Y}} \left[\frac{1}{\underline{K}}\right] = \rho\theta_1 - \dot{\theta}_1$$

$$H_h = \theta_2 \delta (1-u) + \theta_1 (1-\alpha) \underbrace{\underline{K}^{\alpha} (uNh)^{1-\alpha} h_a^{\gamma}}_{\underline{Y}} \left[\frac{1}{h}\right] = \rho\theta_2 - \dot{\theta}_2$$

Note:

Only last condition different when deriving optimal path: $(1-\alpha+\gamma)$ repl. $(1-\alpha)$

Use $H_c = 0$ and $H_K = \rho\theta_1 - \dot{\theta}_1$

$$\alpha \frac{Y}{K} = \rho - \frac{\dot{\theta}_1}{\theta_1} = \rho + \frac{\dot{c}}{c}$$

On BGP: $\frac{\dot{c}}{c}$ constant

$\Rightarrow$ Y, K grow at same rate

$$g_Y = g_K = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K}$$

$$Y = K^{\alpha} N^{1-\alpha} h^{1-\alpha+\gamma} \times (\text{const.})$$

$$g_Y = \alpha g_K + (1-\alpha) n + (1-\alpha+\gamma) v = g_K$$

$$v = \frac{\dot{h}}{h}$$

$$\boxed{g_Y = g_K = \frac{(1-\alpha) n + (1-\alpha+\gamma) v}{(1-\alpha)} = n + \left[\frac{1-\alpha+\gamma}{1-\alpha}\right] v}$$

On BGP, constant saving rate

$$s = 1 - \frac{c}{\left(\frac{Y}{N}\right)} \quad \text{constant}$$

c, $\frac{Y}{N}$ grow at same rate

$$\frac{\dot{c}}{c} = g_c = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} = g_Y - n$$

$$\boxed{g_c = g_Y - n = g_k - n = \left(\frac{1-\alpha+\gamma}{1-\alpha}\right) v}$$

(21) in Lucas

Next: find $v = \frac{\dot{h}}{h}$

Use $H_u = 0$

$$\theta_1 (1-\alpha) \frac{\bar{Y}}{u} = \theta_2 \delta h$$

On BGP, u constant

$$\underbrace{\left(\frac{\dot{\theta}_1}{\theta_1}\right)}_{-g_c} + \underbrace{\left(\frac{\dot{\bar{Y}}}{\bar{Y}}\right)}_{g_{\bar{Y}} = g_c + n} = \frac{\dot{\theta}_2}{\theta_2} + \underbrace{\left(\frac{\dot{h}}{h}\right)}_{v}$$

$$\boxed{n = \frac{\dot{\theta}_2}{\theta_2} + v}$$

Use $H_h = \rho \theta_2 - \dot{\theta}_2$

$$\boxed{\delta (1-u) + \left(\frac{\theta_1}{\theta_2}\right)(1-\alpha)\frac{\bar{Y}}{h} = \rho - \frac{\dot{\theta}_2}{\theta_2}}$$

Use $H_u = 0$

$$\boxed{\frac{\theta_1}{\theta_2} = \frac{\delta h u}{(1-\alpha)\bar{Y}}}$$

$$\delta(1-u) + \underbrace{\left[\frac{\delta h u}{(1-\alpha)\bar{Y}}\right](1-\alpha)\left(\frac{\bar{Y}}{h}\right)}_{\delta u} = \rho - \left(\frac{\dot{\theta}_2}{\theta_2}\right)$$

$$\delta = \rho - \frac{\dot{\theta}_2}{\theta_2} = \rho - [n - v] = \rho + v - n$$

$$\boxed{v = n + \delta - \rho}$$

(26) in Lucas

(II) Optimal Path

All opt. conditions the same except

$$\boxed{H_h = \theta_2 \delta(1-u) + \theta_1(1-\alpha+\gamma)\frac{\bar{Y}}{h} = \rho\theta_2 - \dot{\theta}_2}$$

Still holds that

$$\boxed{n = \frac{\dot{\theta}_2}{\theta_2} + r} \iff \boxed{\frac{\dot{\theta}_2}{\theta_2} = n - r}$$

$$\boxed{\frac{\theta_1}{\theta_2} = \frac{\delta h u}{(1-\alpha)\bar{Y}}}$$

$$\underbrace{\delta(1-u)}_{r} + \left[\frac{\theta_1}{\theta_2}\right](1-\alpha+\gamma)\frac{\bar{Y}}{h} = \rho - \underbrace{\left(\frac{\dot{\theta}_2}{\theta_2}\right)}_{n-r}$$

$$\cancel{v} + \left[\frac{1-\alpha+\gamma}{1-\alpha}\right]\delta u = \rho - [n - \cancel{v}]$$

$$v = \delta(1-u) = \delta - \delta u$$

$$\delta u = \delta - v$$

$$\left[\frac{1-\alpha+\gamma}{1-\alpha}\right](\delta - v) = \rho - n > 0$$

$$v - \delta = -(\rho - n)\left[\frac{1-\alpha}{1-\alpha+\gamma}\right]$$

$$\boxed{v = \delta - (\rho - n)\left[\frac{1-\alpha}{1-\alpha+\gamma}\right]}$$

v^* in (24) in Lucas

$$\nu^* = \underbrace{\delta - (\rho - n)}_{\nu} + \left[\frac{\gamma}{1-\alpha+\gamma}\right](\rho - n)$$

$$\boxed{\nu^* - \nu = \left(\frac{\gamma}{1-\alpha+\gamma}\right)(\rho - n)}$$

Diff. btw. optimal and eq.
growth rate increasing in
importance of externality, γ