Ramsey model with Cobb-Douglas production, isoelastic utility

Dynamical System

$$\boxed{\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[\alpha k^{\alpha-1} - \rho - \theta g\right] \\ \dot{k} &= k^{\alpha} - c - (n+g)k \end{aligned}}$$

Define

$$z = \frac{c}{k}$$

$$x = k^{\alpha-1} = \frac{r}{\alpha}$$

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$$

$$\frac{\dot{k}}{k} = \underbrace{k^{\alpha - 1}}_{x} - \underbrace{\left(\frac{c}{k}\right)}_{z} - (n+g)$$

$$\frac{\dot{z}}{z} = \frac{1}{\theta}\left[\alpha x - \rho - \cancel{\theta g}\right] - x + z + (n + \cancel{g})$$

$$\boxed{\frac{\dot{z}}{z} = \frac{1}{\theta}\left[(\alpha - \theta)x - \rho + \theta z + \theta n\right]}$$

$$\frac{\dot{x}}{x} = \frac{\partial}{\partial t}\left[\ln\left(k^{\alpha-1}\right)\right]$$

$$= -(1-\alpha)\frac{\dot{k}}{k}$$

$$\boxed{\frac{\dot{x}}{x} = -(1-\alpha)\left[x - z - (n+g)\right]}$$

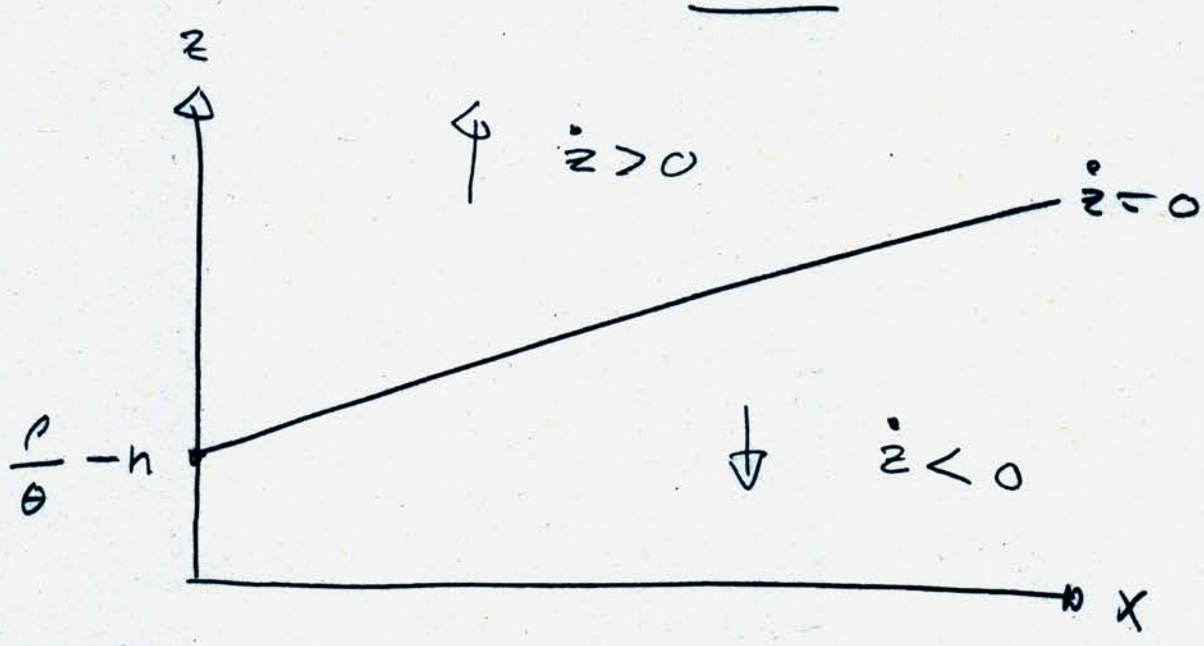
Phase Diagram

$(\dot{z}=0)$ - locus

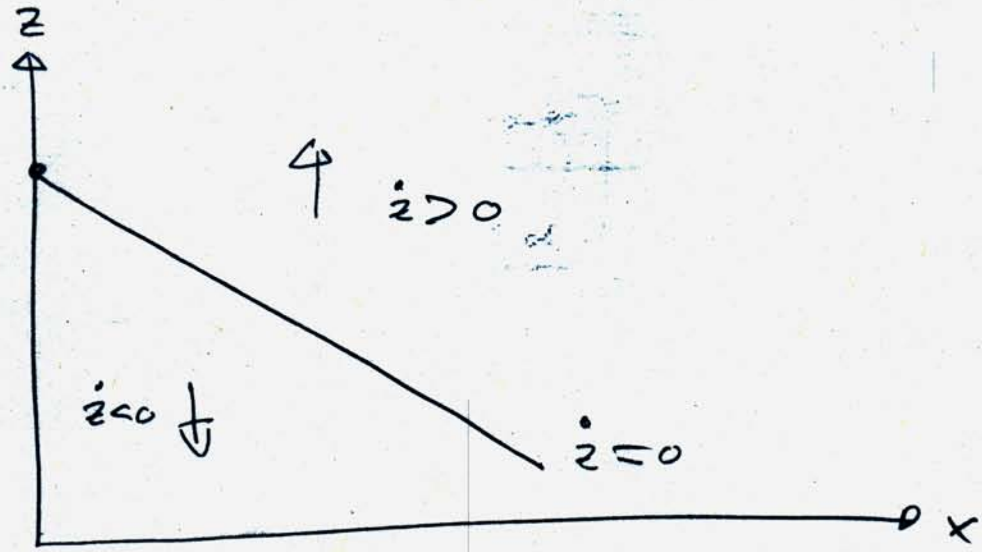
$$\theta z + \theta n = \rho + (\theta - \alpha) x$$

$$\boxed{z = \frac{\rho}{\theta} - n + \underbrace{\left(\frac{\theta-\alpha}{\theta}\right)}_{<1} x}$$

$\theta > \alpha$

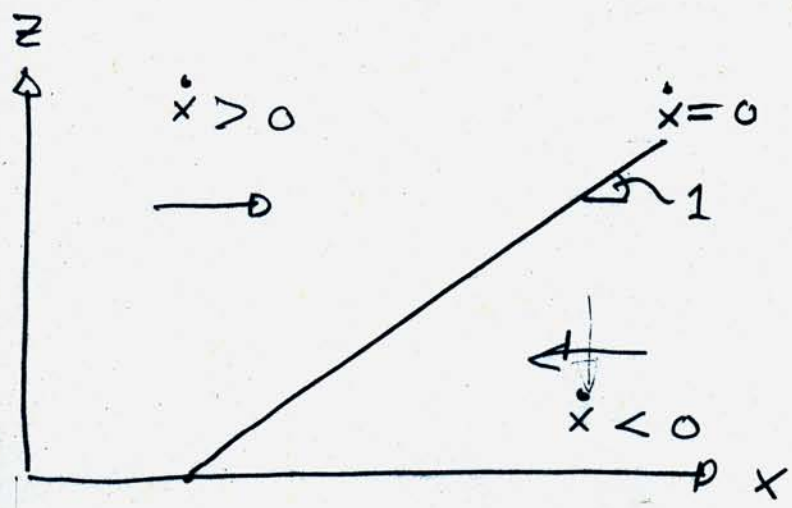


$0 < \alpha$

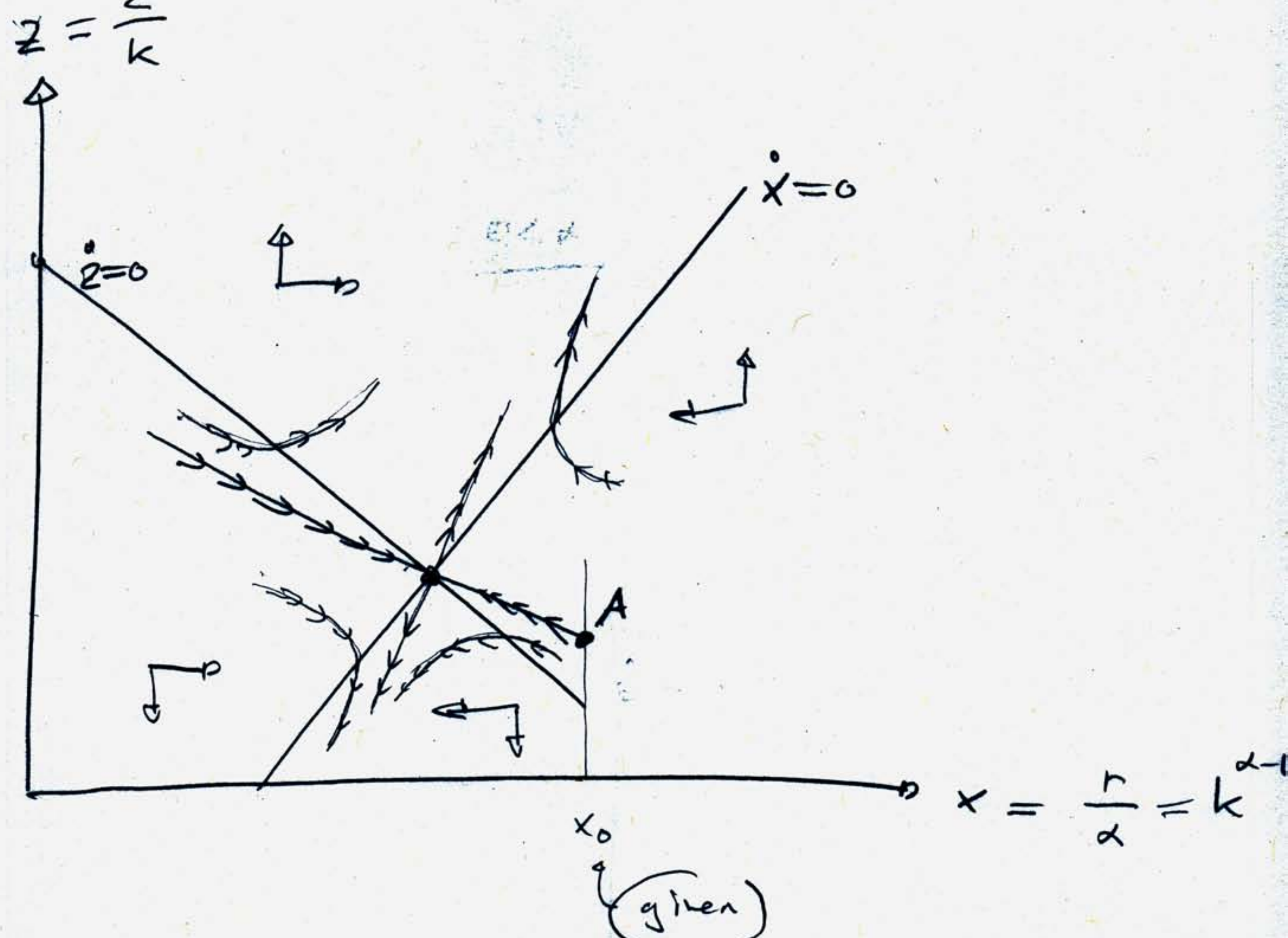


$(\dot{x} = 0)$ - locus

$$\boxed{z = -(n+g) + x}$$



$\theta < \alpha$

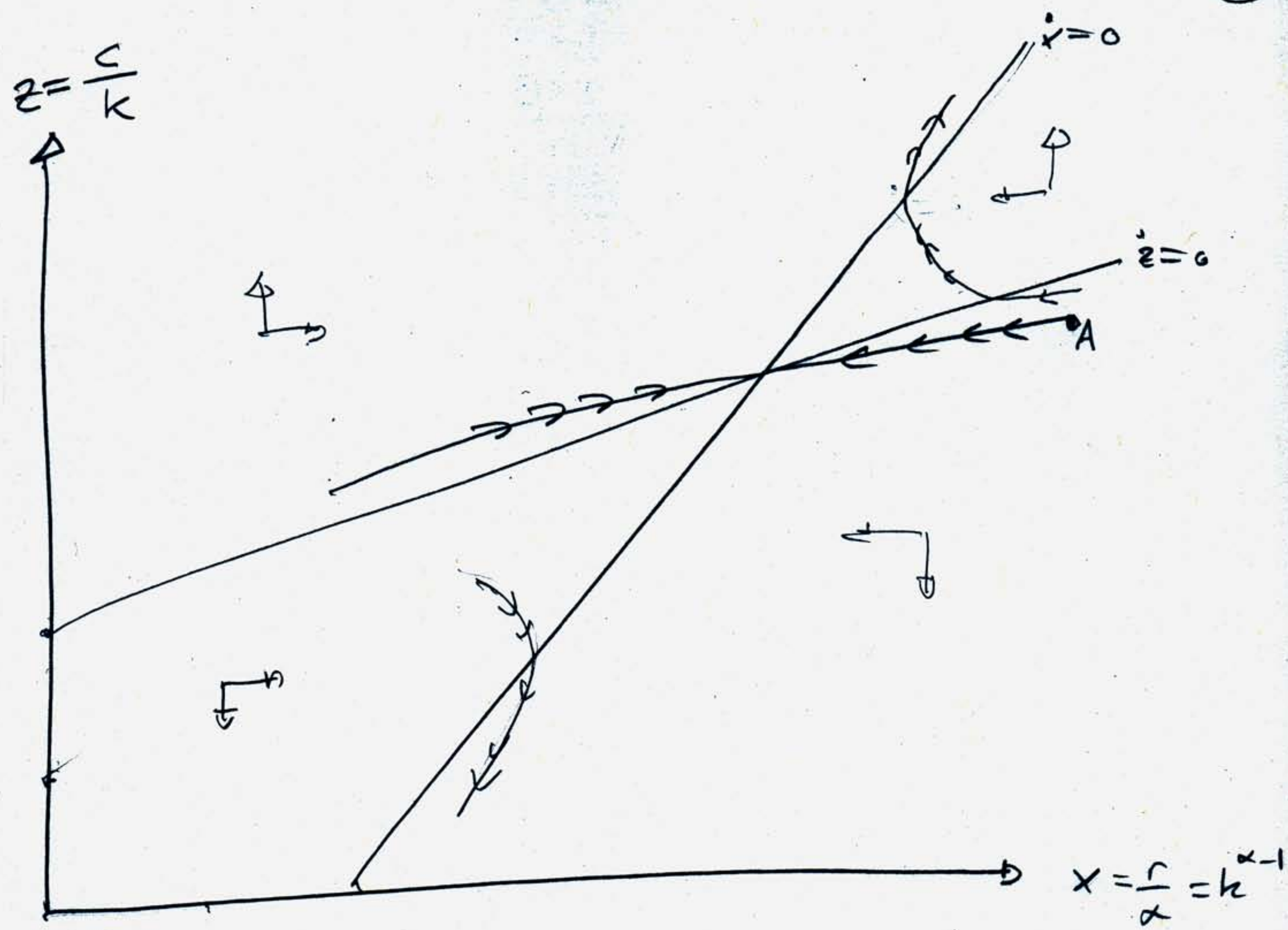


Start at A:

* Saddle path with decreasing r (increasing k), and increasing z

* $$\frac{z}{x} = \frac{\frac{c}{k}}{k^{\alpha-1}} = \frac{c}{k^{\alpha}} = 1 - s$$

$$\frac{z\uparrow}{x\downarrow} = 1 - s\downarrow$$; s decreasing



<u>Start at A</u>

* Saddle Path with decreasing x, decreasing z

* Effect on $1 - s = \frac{z}{x}$ ambiguous

Steady State

$$z = -(n+g) + x \qquad \text{from } \dot{x}=0$$

$$z = \left(\frac{\rho}{\theta} - n\right) + \left(\frac{\theta - \alpha}{\theta}\right) x \qquad \text{from } \dot{z}=0$$

$$-(n+g) + x = \left(\frac{\rho}{\theta} - n\right) + \left(1 - \frac{\alpha}{\theta}\right) x$$

$$-g = \frac{\rho}{\theta} - \left(\frac{\alpha}{\theta}\right) x$$

$$\boxed{x = \frac{1}{\alpha}\left[\rho + \theta g\right] = k^{\alpha - 1} = \frac{r}{\alpha}}$$

$$r = \rho + \theta g$$

$$z = \frac{c}{k} = -(n+g) + \frac{1}{\alpha}(\rho + \theta g)$$

$$\boxed{\begin{gathered} c = \underbrace{\left(\frac{k}{\alpha}\right)(\rho + \theta g)} - (n+g)k \stackrel{ls}{=} k^{\alpha} - (n+g)k \\ xk = k^{\alpha} \end{gathered}}$$

$$\alpha k^{\alpha - 1} = \rho + \theta g = r$$

$$k^{\alpha - 1} = \frac{\rho + \theta g}{\alpha}$$

$$k = \left(\frac{\alpha}{\rho + \theta g}\right)^{\frac{1}{1-\alpha}}$$

$$\boxed{c = \left[\frac{\alpha}{\rho + \theta g}\right]^{\frac{\alpha}{1-\alpha}} - (n+g)\left[\frac{\alpha}{\rho + \theta g}\right]^{\frac{1}{1-\alpha}}}$$