Pontryagin's Maximum Principle

Problem: choose $u(t)$, $x(t)$ for $t \in [0, T]$

to maximize

$$\int_0^T f(t, x(t), u(t))\,dt$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t))$$

$x(0)$, $x(T)$ given

Idea: Set up Lagranian

Infinitely many constraints, Lagranian multipliers

$$\mathcal{L} = \int_0^T f(t, x(t), u(t))\,dt + \int_0^T \lambda(t)\left[g(t, x(t), u(t)) - \dot{x}(t)\right]dt$$

Can be shown that the optimal functions $u(t)$, $x(t)$: satisfy (suppressing t's)

$$\boxed{\begin{aligned} &\frac{d}{du}\left[f + \lambda g\right] = 0 \\ &\frac{d}{dx}\left[f + \lambda g\right] = -\dot{\lambda} \end{aligned}}$$

These are <u>the optimality conditions</u> and that they give the solution to the max problem is called

<u>Pontryagin's Max. Principle</u>

Recall: <u>integration by parts</u>

Let $F(x)$, $G(x)$ be functions with derivatives

$$F'(x) = f(x), \quad G'(x) = g(x)$$

and let $H(x) = F(x)\,G(x)$

Chain rule:

$$H'(x) = f(x)\,G(x) + F(x)\,g(x)$$

$$\int_a^b H'(x)\,dx = H(b) - H(a)$$

$$= \underbrace{F(b)\,G(b)}_{H(b)} - \underbrace{F(a)\,G(a)}_{H(a)}$$

$$H(b) - H(a) = \int_a^b f(x)\,G(x)\,dx + \int_a^b F(x)\,g(x)\,dx$$

$$\boxed{-\int_a^b F(x)\,g(x)\,dx = \int_a^b f(x)\,G(x)\,dx + H(a) - H(b)}$$

It follows that:

$$\boxed{-\int_0^T \lambda(t)\,\dot{x}(t)\,dt = \int_0^T \dot{\lambda}(t)\,x(t)\,dt + \lambda(0)\,x(0) - \lambda(T)\,x(T)}$$

$$\left(\begin{array}{lcl} \lambda(t) & \text{corresponds to} & F(x) \\ x(t) & \text{——}''\text{——} & G(x) \end{array}\right)$$

Substitute into Lagrangian

$$\mathcal{L} = \int_0^T f(t, x, u)\,dt + \int_0^T \lambda\left[g(t, x, u) - \dot{x}\right]dt$$

$$= \int_0^T \left[f + \lambda g\right]dt - \int_0^T \lambda \dot{x}\,dt$$

$$= \int_0^T \left[f + \lambda g + \dot{\lambda} x\right]dt + \underbrace{\lambda(0)\,x(0) - \lambda(T)\,x(T)}_{\text{given}}$$

$f + \lambda g$ often referred to as Hamiltonian

<u>Some "cheating"</u>: think of integral as (discrete) sum and disregard "additive" elements

$$\boxed{\begin{aligned} &\frac{d}{du}\left[\int_0^T [f + \lambda g + \dot{\lambda} x]\, dt\right] \\ &= \frac{d}{du}\left[f + \lambda g\right] = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\frac{d}{dx}\left[\int_0^T [f + \lambda g + \dot{\lambda} x]\, dt\right] \\ &= \frac{d}{dx}\left[f + \lambda g\right] + \dot{\lambda} = 0 \end{aligned}}$$

Ramsey Model

Here: no pop. growth, techn. change

$$\max \int_0^\infty e^{-\rho t} u(c(t))\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = w(t) + r(t)k(t) - c(t)$$

Set up Hamiltonian

(Here: present-value)

$$H(t, c(t), k(t)) = e^{-\rho t} u(c(t)) + \lambda(t)\left[w(t) + r(t)\,k(t) - c(t)\right]$$

Optimality conditions

$$H_c(t, c(t), k(t)) = 0$$

$$H_k(t, c(t), k(t)) = -\dot{\lambda}(t)$$

$$\boxed{\begin{aligned} e^{-\rho t} u'(c(t)) &= \lambda(t) \\ \lambda(t)\, r(t) &= -\dot{\lambda}(t) \end{aligned}}$$

$(H_c = 0)$

$(H_k = -\dot{\lambda})$

Take logs, derivative of $H_c = 0$

$$-\rho t + \ln[u'(c)] = \ln(\lambda)$$

$$-\rho + \frac{\dot{c}\, u''(c)}{u'(c)} = \frac{\dot{\lambda}}{\lambda} = -r$$

$$\boxed{\frac{\dot{c}}{c} = \left[\frac{-u'(c)}{u''(c)\, c}\right](r - \rho)}$$

Euler Equation

If $u(c) = \dfrac{c^{1-\theta}}{1-\theta}$, then

$$\frac{-u''(c)\,c}{u'(c)} = \theta$$

Euler Eq. becomes:

$$\boxed{\frac{\dot{c}}{c} = \frac{1}{\theta}\left[r - \rho\right]}$$

Comparison to discrete-time Hamiltonian (or Lagrangian)

(β replacing α in notes ; $n = 1$)

$$\boxed{\begin{aligned} \beta^t u'(c_t) &= \lambda_{t+1} \qquad (*) \\ \lambda_{t+1}(1+r_t) &= \lambda_t \qquad (**) \end{aligned}}$$

"Translate" to continuous time

β	corresponds to	$e^{-\rho}$
c_t	—— " ——	$c(t)$
λ_t, λ_{t+1}	—— " ——	$\lambda(t)$
$\lambda_{t+1} - \lambda_t$	—— " ——	$\dot{\lambda}(t)$

(*) becomes $e^{-\rho t} u'(c(t)) = \lambda(t)$

(**) becomes $\lambda(t) r(t) = -\dot{\lambda}(t)$