# Pontryagin's maximum principle

Problem: for $t \in [0, T]$, choose $u(t)$, $x(t)$ to maximize

$$\int_0^T f(t, x(t), u(t))dt$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t))$$

and $x(0)$, $x(T)$ given

That is: find optimal paths $\{u(t), x(t)\}_{t=0}^T$

## Illustration

Start with $x(0)$. What is $x(\Delta)$ for some small $\Delta > 0$?

$$
\begin{aligned}
x(\Delta) &\approx x(0) + \dot{x}(0)\Delta \\
&= x(0) + g(0, x(0), u(0))\Delta
\end{aligned}
$$

Choice of $u(0)$ determines $x(\Delta)$

Choice of $u(\Delta)$ determines $x(2\Delta)$

Choice of $u(2\Delta)$ determines $x(3\Delta)$, and so on

Discrete jumps of length $\Delta$

Illustrates why $u(t)$ called *control variable*, $x(t)$ the *state variable*

**Slightly more formal approach**

Set up Lagrangian

$$\mathcal{L} = \int_0^T f(t, x(t), u(t))dt + \int_0^T \lambda(t) \left[g(t, x(t), u(t)) - \dot{x}(t)\right] dt$$

One Lagrangian multiplier $\lambda(t)$ for each constraint $\dot{x}(t) = g(t, x(t), u(t))$

Recall integration by parts:

$$\frac{\partial \left[x(t)\lambda(t)\right]}{\partial t} = \lambda(t)\dot{x}(t) + \dot{\lambda}(t)x(t)$$

$$
\begin{aligned}
x(T)\lambda(T) - x(0)\lambda(0) &= \int_0^T \frac{\partial\left[x(t)\lambda(t)\right]}{\partial t} dt \\
&= \int_0^T \left[\lambda(t)\dot{x}(t) + \dot{\lambda}(t)x(t)\right] dt \\
&= \int_0^T \lambda(t)\dot{x}(t)dt + \int_0^T \dot{\lambda}(t)x(t)dt
\end{aligned}
$$

$$
\int_0^T \lambda(t)\dot{x}(t)dt = x(T)\lambda(T) - x(0)\lambda(0) - \int_0^T \dot{\lambda}(t)x(t)dt
$$

Suppress arguments, substitute back into Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= \int_0^T \left[f + \lambda g - \lambda \dot{x}\right] dt \\
&= \int_0^T [f + \lambda g]\, dt - \int_0^T \lambda \dot{x} dt \\
&= \int_0^T [f + \lambda g]\, dt - \underbrace{\left(x(T)\lambda(T) - x(0)\lambda(0) - \int_0^T \dot{\lambda} x dt\right)}_{=\int_0^T \lambda \dot{x} dt} \\
&= \int_0^T \left[f + \lambda g + \dot{\lambda} x\right] dt + \underbrace{x(0)\lambda(0) - x(T)\lambda(T)}_{\text{given}}
\end{aligned}
$$

Consider optimal choice over interval $[\tau, \tau + \Delta] \subseteq [0, T]$, for some $\tau \in [0, T)$ and $\Delta > 0$

Intuition: "chop up" the integral at $\tau$

$$\begin{aligned}\mathcal{L} \;=\; & \int_0^\tau \left[f + \lambda g + \dot{\lambda} x\right] dt + \int_{\tau+\Delta}^T \left[f + \lambda g + \dot{\lambda} x\right] dt \\ & + x(0)\lambda(0) - x(T)\lambda(T) \\ & + \int_\tau^{\tau+\Delta} \left[f + \lambda g + \dot{\lambda} x\right] dt\end{aligned}$$

Focus on last term; rest additive constants

Divide by the constant $\Delta$

$$\frac{1}{\Delta}\int_{\tau}^{\tau+\Delta}\left[f+\lambda g+\dot{\lambda}x\right]dt$$

$$\approx \underbrace{f(\tau,x(\tau),u(\tau))+\lambda(\tau)g(\tau,x(\tau),u(\tau))}_{\equiv H(\tau,x(\tau),u(\tau),\lambda(\tau))}+\dot{\lambda}(\tau)x(\tau)$$

Note: approximation error vanishes as $\Delta$ approaches zero

Why? If $G'(x)=g(x)$, then

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{a}^{a+\Delta}g(x)dx=\lim_{\Delta\to 0}\frac{G(a+\Delta)-G(a)}{\Delta}=g(a)$$

(Recall definition of derivative)

The first-order conditions associated with the Lagrangian:

$$\frac{\partial H(\tau, x(\tau), u(\tau), \lambda(\tau))}{\partial u(\tau)} = 0$$

$$\frac{\partial H(\tau, x(\tau), u(\tau), \lambda(\tau))}{\partial x(\tau)} = -\dot{\lambda}(\tau)$$

Must hold for all $\tau \in [0, T]$

Just change time indexation:

$$\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial u(t)} = 0 \qquad (*)$$

$$\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x(t)} = -\dot{\lambda}(t) \qquad (**)$$

$u(t)$ called *control variable*, $x(t)$ the *state variable*, $\lambda(t)$ the *costate variable*

$H(t, x(t), u(t), \lambda(t))$ called the *Hamiltonian*

(*) and (**) are the *optimality conditions* associated with that Hamiltonian

These conditions are *necessary* for $u(t)$, $x(t)$ to be optimal for all $t \in [0, T]$

That is: the optimal paths $\{u(t), x(t)\}_{t=0}^{T}$ must satisfy (*) and (**)

Must also satisfy *Transversality condition*

Can be derived from $\dot{x}(t) = g(t, x(t), u(t))$

See application to Ramsey model in Problem 5 under "More problems" at Econ 5011 website