Ramsey with population growth and technological progress (Acemoglu, Sec. 8.6)

$L(t)$ = work force

$A(t)$ = level of technology

= labor-augmenting technology

$$\dot{L}(t) = n L(t) \Rightarrow L(t) = L(0) e^{nt}$$

$$\dot{A}(t) = g A(t) \Rightarrow A(t) = A(0) e^{gt}$$

$C(t)$ = consumption per worker

$$c(t) = \frac{C(t)}{A(t)}$$

$$U = \int_0^\infty e^{-\rho t} u(C) L(t)\, dt$$

Isoelastic utility: $u(C) = \dfrac{C^{1-\theta}}{1-\theta}$

$$U = \int_0^\infty e^{-\rho t} \frac{[A(t)c(t)]^{1-\theta}}{1-\theta} L(t)\, dt$$

$$U = L(0)[A(0)]^{1-\theta} \int_0^\infty e^{-[\rho - n - (1-\theta)g]t} \frac{[c(t)]^{1-\theta}}{1-\theta}\, dt$$

Let $L(0) = A(0) = 1$

$$\beta = \rho - n - (1-\theta)g$$

Assume $\rho > n + (1-\theta)g \iff \beta > 0$

$$U = \int_0^\infty e^{-\beta t} \frac{[c(t)]^{1-\theta}}{1-\theta}$$

Budget constraint

$\bar{K}(t)$ = total capital stock

$$\dot{\bar{K}}(t) = w(t)A(t)L(t) + r(t)\bar{K}(t) - \underbrace{C(t)}_{=A(t)c(t)}L(t)$$

$$k(t) = \frac{\bar{K}(t)}{A(t)L(t)}$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{\bar{K}}(t)}{\bar{K}(t)} - (n+g)$$

$$\frac{\dot{\bar{K}}(t)}{\bar{K}(t)} = \frac{w(t)}{k(t)} + r(t) - c(t)\frac{A(t)L(t)}{\bar{K}(t)}$$

$$\frac{\dot{k}(t)}{k(t)} = \underbrace{\frac{1}{k(t)}\left[w(t) + r(t)k(t) - c(t)\right]}_{\dot{\bar{K}}(t)/\bar{K}(t)} - (n+g)$$

$$\boxed{\dot{k}(t) = w(t) + r(t)k(t) - c(t) - (n+g)k(t)}$$

$$\boxed{\bar{U} = \int_0^\infty e^{-\beta t} \frac{[c(t)]^{1-\theta}}{1-\theta} \, dt}$$

Present-value Hamiltonian

$$H(k(t), c(t), \lambda(t)) = e^{-\beta t} \frac{[c(t)]^{1-\theta}}{1-\theta}$$

$$+ \lambda(t) \left[w(t) + r(t)k(t) - c(t) - (n+g)k(t) \right]$$

$$H_c(\cdot) = e^{-\beta t} (c(t))^{-\theta} - \lambda(t) = 0$$

$$H_k(\cdot) = \lambda(t) \left[r(t) - (n+g) \right] = - \dot{\lambda}(t)$$

Finding Euler equation

$$e^{-\beta t}\left[c(t)\right]^{-\theta} = \lambda(t)$$

Take logs, time derivative

$$-\beta - \theta\left[\frac{\dot{c}(t)}{c(t)}\right] = \frac{\dot{\lambda}(t)}{\lambda(t)}$$

Use $H_k(\cdot) = 0$, $\beta = \rho - n - (1-\theta)g$

$$-\left[\rho - n - (1-\theta)g\right] - \theta\frac{\dot{c}(t)}{c(t)} = (n+g) - r(t)$$

$$\boxed{\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left[r(t) - \rho - \theta g\right]}$$

Dynamical system

$$\dot{c}(t) = \frac{c(t)}{\theta}\left[f'(k(t)) - \rho - \theta g\right]$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

Recall: $w(t) = f(k(t)) - f'(k(t))k(t)$

$r(t)k(t) = f'(k(t))k(t) \quad (\delta = 0)$

Transversality condition

$$\lim_{t \to \infty} \lambda(t)k(t) = 0$$

Phase Diagram

$\dot{c}(t) = 0$ when $k(t) = k^*$

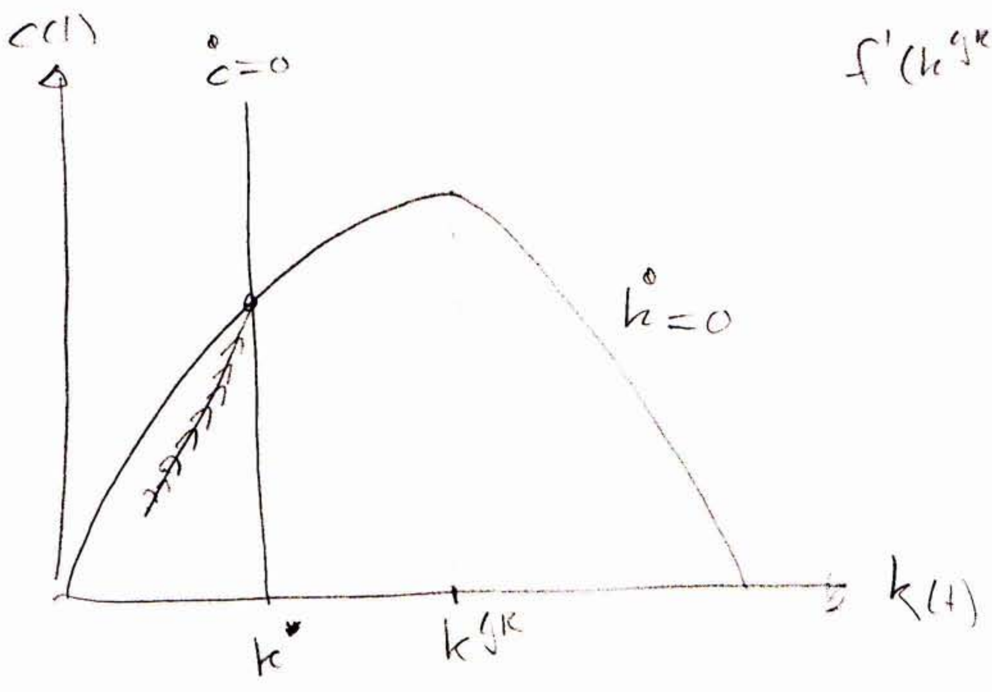
$$f'(k^*) \equiv \rho + \theta g$$

$\dot{k}(t) = 0$ when $c(t) = \phi(k(t))$

$$\phi(k) = f(k) - (n+g)k$$

$$\phi'(k) = f'(k) - (n+g)$$

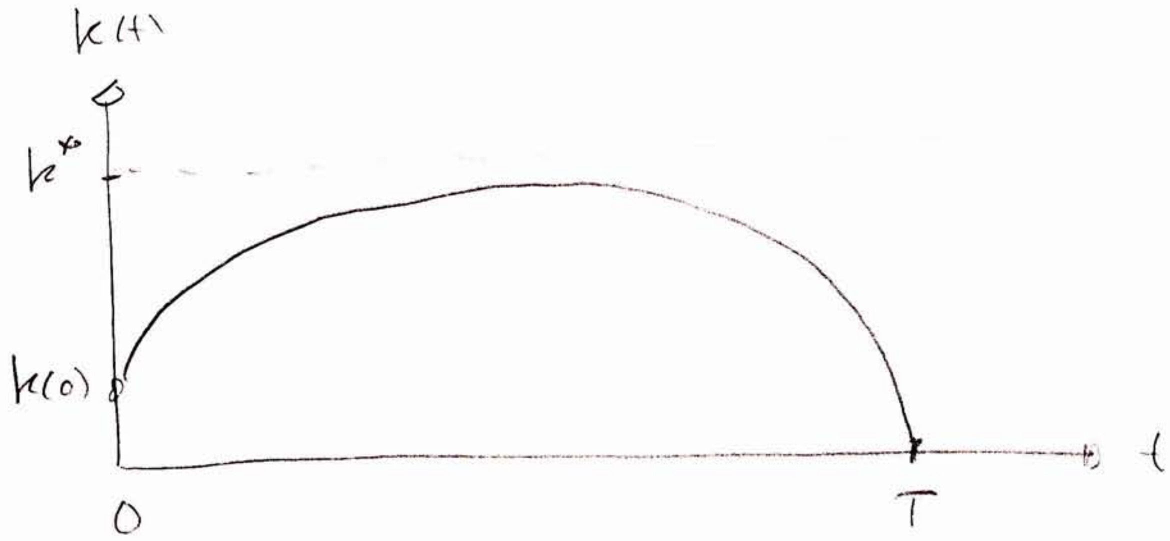
$$\phi''(k) = f''(k) < 0$$

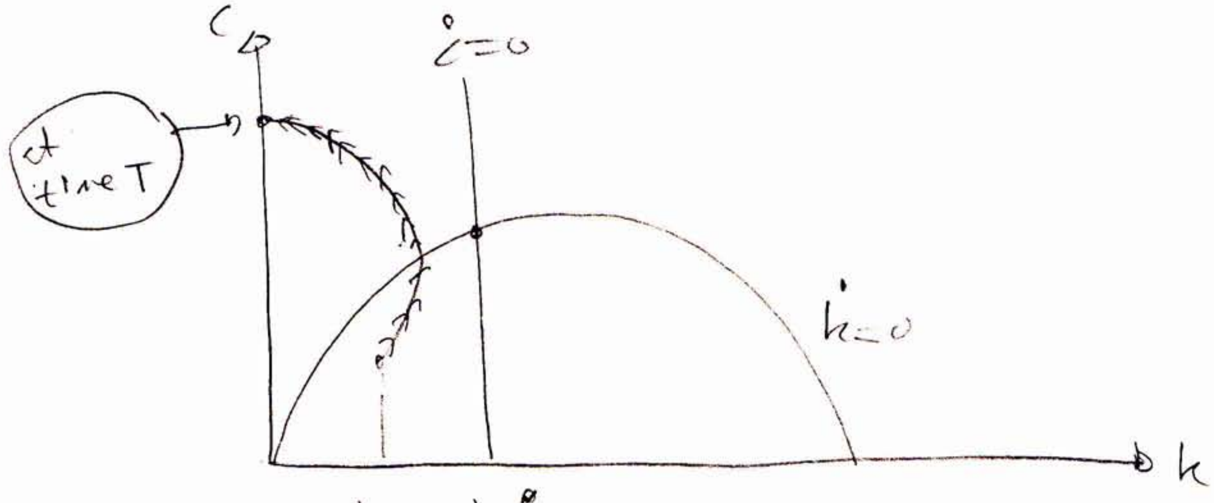


$f'(k^{gr}) \equiv n+g$

Understanding transversality condition

Finite horizon : $k(T) = 0$





Path → saddle path as $T \to \infty$

(cf. Problem 8, Econ 5011 Probl. Set)