Econ 5700 slides
Growth models: Solow, Diamond, Malthus

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Basics: terminology etc.

Growth models are dynamic models: involve time, t

Here focus on *discrete-time* models: the variable t (time) is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Variables usually written x_t

Not "$x(t)$" which is typically continuous time notation

Discrete-time variable cannot be differentiated with respect to t

Dynamics described by difference equations: $x_{t+1} = \phi(x_t)$

Difference equations

Mostly on the form $x_{t+1} = \phi(x_t)$

$\overline{x}$ is a steady state (or steady-state equilibrium) to $x_{t+1} = \phi(x_t)$ if, and only if, $\overline{x} = \phi(\overline{x})$

Means that $\overline{x}$ is a *fixed point* to $\phi(x)$

Stability

$\overline{x}$ is locally stable if, and only if, $\phi'(\overline{x}) \in (-1, 1)$

$\overline{x}$ is globally stable if x_t converges to $\overline{x}$ regardless of starting value, x_0

- Usually equivalent to $\overline{x}$ being (a) locally stable, and (b) unique

If $\phi'(\overline{x}) < 0$ the steady state is called *oscillatory*

Locally stable steady state: $\phi'(\overline{x}) \in (0, 1)$

Locally unstable steady state: $\phi'(\overline{x}) > 1$

Oscillatory locally stable steady state: $\phi'(\overline{x}) \in (-1, 0)$

Oscillatory locally non-stable steady state: $\phi'(\overline{x}) < -1$

Production functions

Standard setting: Y=output, K (capital) and L (labor)

K, L referred to as inputs

$$Y = F(K, L) \tag{1}$$

Usually assumed to satisfy:

(1) Positive marginal products:

$$F_K(\cdot) > 0, F_L(\cdot) > 0 \tag{2}$$

(2) Diminishing marginal products:

$$F_{KK}(\cdot) < 0, F_{LL}(\cdot) < 0 \tag{3}$$

(3) The Inada condition:

$$\begin{aligned} \lim_{Z \to 0} F_Z(\cdot) &= \infty \\ \lim_{Z \to \infty} F_Z(\cdot) &= 0 \end{aligned} \quad (4)$$

for $Z = K, L$

(4) Constant Returns to Scale (CRS):

$$\lambda F(K, L) = F(\lambda K, \lambda L) \quad (5)$$

for all $\lambda > 0$

Intensive-form production function

Let lower-case variables denote per-worker levels

CRS implies

$$y = \frac{Y}{L} = \frac{F(K,L)}{L} = F\left(\frac{K}{L}, 1\right) = F(k,1) \equiv f(k) \tag{6}$$

Assumptions (1)-(3) imply:

$$\begin{aligned} f'(k) &> 0 \\ f''(k) &< 0 \\ \lim_{k \to 0} f'(k) &= \infty \\ \lim_{k \to \infty} f'(k) &= 0 \end{aligned} \tag{7}$$

Also, using l'Hôpital's Rule:

$$\lim_{k \to 0} \frac{f(k)}{k} = \lim_{k \to 0} \frac{f'(k)}{1} = \infty \tag{8}$$

Factor prices

Atomistic firms take factor prices as given when maximizing profits:

$$\max_{K,L} F(K,L) - \delta K - wL - rK \tag{9}$$

w=wage rate, r=real interest rate, δ=depreciation rate

$F(K,L) - \delta K$=net output (output net of capital depreciation)

Rewrite $F(K,L) = Lf(k) = Lf(\frac{K}{L})$

Exercise: show that

$$\begin{aligned} w &= F_L(K,L) = f(k) - f'(k)k \\ r &= F_K(K,L) - \delta = f'(k) - \delta \end{aligned} \tag{10}$$

Parametric examples of production functions

Cobb-Douglas:

$$\begin{aligned} F(K,L) &= K^{\alpha}L^{1-\alpha} \\ f(k) &= k^{\alpha} \end{aligned} \tag{11}$$

where $\alpha \in (0,1)$

CES (various formulations):

$$\begin{aligned} F(K,L) &= [\alpha K^{\sigma} + (1-\alpha)L^{\sigma}]^{\frac{1}{\sigma}} \\ f(k) &= [\alpha k^{\sigma} + (1-\alpha)]^{\frac{1}{\sigma}} \end{aligned} \tag{12}$$

where $\alpha \in (0,1)$ and $\sigma \in (-\infty, 1]$, $\sigma \neq 0$

Note: CES does not always satisfy Inada

The Solow Growth Model

Discrete time setting: the time variable t is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Notation:

$K_t =$ capital in period t

$\delta =$ depreciation rate

$s =$ rate of saving/investment out of income, Y_t

Evolution of capital stock:

$$K_{t+1} = sY_t + (1 - \delta)K_t \tag{13}$$

L_t = population/labor force in period t

n = (net) growth rate of population

$$L_{t+1} = (1 + n)L_t \tag{14}$$

Assume $n > 0$, $\delta \in (0, 1]$; $s \in (0, 1]$

$$Y_t = F(K_t, L_t) \tag{15}$$

Task: find difference equation for $k_t = K_t/L_t$, on the form: $k_{t+1} = \phi(k_t)$

Use (13) to (15), and $y_t = Y_t/L_t = F(k_t, 1) = f(k_t)$, to get

$$\begin{aligned} \tfrac{K_{t+1}}{L_t} &= \tfrac{K_{t+1}}{L_{t+1}}\tfrac{L_{t+1}}{L_t} = k_{t+1}(1+n) \\ &= \tfrac{sY_t+(1-\delta)K_t}{L_t} = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t \end{aligned} \tag{16}$$

Or:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{17}$$

Properties of $\phi(k_t)$

$$\phi'(k_t) = \frac{1-\delta}{1+n} + \frac{s}{1+n} f'(k_t) > 0$$

$$\phi''(k_t) = \frac{s}{1+n} f''(k_t) < 0$$

$$\lim_{k_t \to 0} \phi'(k_t) = \frac{1-\delta}{1+n} + \frac{s}{1+n} \lim_{k_t \to 0} f'(k_t) = \infty \tag{18}$$

$$\lim_{k_t \to \infty} \phi'(k_t) = \frac{1-\delta}{1+n} + \frac{s}{1+n} \lim_{k_t \to \infty} f'(k_t) = \frac{1-\delta}{1+n} < 1$$

Together these guarantee: existence, uniqueness, and stability of steady state

(Uniqueness except for the trivial one where $k_t = 0$)

Illustrate in 45°-diagram

Check: If one of the 4 properties above change, how come existence, uniqueness, and stability are no longer guaranteed?

Problems: relaxing assumption (3); and (b) letting saving rate (s) differ across factor payments

Parametric example: Cobb-Douglas production

$$\begin{aligned} Y_t &= Z K_t^{\alpha} L_t^{1-\alpha} \\ y_t &= Z k_t^{\alpha} \end{aligned} \tag{19}$$

$$k_{t+1} = \frac{sZk_t^{\alpha} + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{20}$$

Steady state capital stock per worker:

$$\overline{k} = \left(\frac{sZ}{n+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{21}$$

Steady state output per worker:

$$\overline{y} = Z\left(\frac{sZ}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} = Z^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{22}$$

Illustrate dynamics, steady state in 45°-diagram

What is effect on output per worker from rise in productivity, Z? Steady state and transition? (Let $Z = Z^{*}$ for $t < \widehat{t}$ and $Z = Z^{**} > Z^{*}$ for $t \geq \widehat{t}$.)

The Diamond Overlapping Generations Model

Agents live in two periods: working age, retirement

L_t = number young (working) agents in period t; $L_{t+1} = (1+n)L_t$

$c_{1,t}$ = consumption of working agent in period t

$c_{2,t}$ = consumption of retired agent in period t

s_t = saving of working agent in period t

$R_{t+1} = 1 + r_{t+1}$ = gross interest rate on savings held from period t to $t+1$

w_t = period-t wage rate

Consider agent young/working in period t

Budget constraints

$$c_{1,t} = w_t - s_t \tag{23}$$

$$c_{2,t+1} = R_{t+1} s_t \tag{24}$$

Utility:

$$U_t = U(c_{1,t}, c_{2,t+1}) \tag{25}$$

Optimal savings decision given by $s(w_t, R_{t+1})$, defined from

$$s(w_t, R_{t+1}) = \underset{s_t \in [0, w_t]}{\arg\max} U(w_t - s_t, R_{t+1} s_t) \tag{26}$$

Capital accumulation:

Total savings in period t = total capital stock in period $t+1$

$$s(w_t, R_{t+1})L_t = K_{t+1} \tag{27}$$

Recall: $L_{t+1} = (1+n)L_t$

$$\frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) = s(w_t, R_{t+1}) \tag{28}$$

Factor prices (recall) given by marginal products

$$w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t) \tag{29}$$

$$R_{t+1} = f'(k_{t+1}) + 1 - \delta \equiv R(k_{t+1}) \tag{30}$$

Thus:

$$k_{t+1} = \phi(k_t) \tag{31}$$

where $\phi(k_t)$ is defined from

$$\phi(k_t) = \frac{s\{w(k_t), R(\phi(k_t))\}}{1+n} \tag{32}$$

The very general case: possibility of indeterminacies

At given k_t, k_{t+1} is not necessarily unique. To see this, hold k_t constant, and replace $\phi(k_t)$ in (32) with k_{t+1}, i.e., $k_{t+1} = s\{w(k_t), R(k_{t+1})\}/(1+n)$.

Always true that $R'(k_{t+1}) < 0$. Assume $\partial s(\cdot)/\partial R_{t+1} < 0$ (over some interval). Then the right-hand side is increasing in k_{t+1}. Since the left-hand side is too, there could be multiple k_{t+1} for given k_t

Intuition: if savings are high, interest rates are low (=marginal product of capital), sustaining high savings [given $\partial s(\cdot)/\partial R_{t+1} < 0$]

Called an *indeterminacy;* implies backward bending $\phi(k_t)$

Illustration: see 45°-diagram

Parametric example: logarithmic utility, Cobb-Douglas production

Log utility:

$$U_t = (1 - \beta)\ln(c_{1,t}) + \beta \ln(c_{2,t+1}) \tag{33}$$

First-order condition:

$$-(1 - \beta)(w_t - s_t)^{-1} + \beta s_t^{-1} = 0 \tag{34}$$

Solving for s_t:

$$s_t = \beta w_t \tag{35}$$

Cobb-Douglas production:

$$\begin{aligned} Y_t &= Z K_t^{\alpha} L_t^{1-\alpha} \qquad (36) \\ y_t &= Z k_t^{\alpha} \\ w_t &= (1 - \alpha) Z k_t^{\alpha} \end{aligned}$$

$$k_{t+1} = \frac{\beta(1-\alpha)Zk_t^{\alpha}}{1+n} = \phi(k_t) \tag{37}$$

Steady state capital stock per worker:

$$\overline{k} = \left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{1}{1-\alpha}} \tag{38}$$

Steady state output per worker:

$$\overline{y} = Z\left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{\alpha}{1-\alpha}} = Z^{\frac{1}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+n}\right]^{\frac{\alpha}{1-\alpha}} \tag{39}$$

Illustrate dynamics, steady state in 45°-diagram; no indeterminacies

What is the effect on output per worker from rise in productivity, Z? Steady state and transition? (Let $Z = Z^{*}$ for $t < \widehat{t}$ and $Z = Z^{**} > Z^{*}$ for $t \geq \widehat{t}$.)

Poverty traps in the Diamond OLG model

Keep assumption about logarithmic utility ($s_t = \beta w_t$) but allow for richer production function

$$k_{t+1} = \frac{\beta \left[f(k_t) - f'(k_t) k_t \right]}{1+n} \equiv \phi(k_t) \tag{40}$$

$$\phi'(k_t) = \frac{-\beta}{1+n} f''(k_t) k_t > 0 \tag{41}$$

$$\phi''(k_t) = \frac{-\beta}{1+n} \left[f'''(k_t) k_t + f''(k_t) \right] \gtrless 0 \tag{42}$$

The sign of $\phi''(k_t)$ depends on the *third derivative* of production function, about which we have not made any assumptions

Multiple steady states possible, even with neoclassical production function

Illustrate dynamics, (all) steady states in 45°-diagram

Which are stable, unstable?

The Malthus Model

Agents live in two periods: children, adults

$L_t =$ number adult (working) agents in period t

$c_t =$ consumption of adult agent in period t

$q =$ cost per child (can be interpreted as consumption per child); exogenous

$n_t =$ number of children per adult

$$L_{t+1} = n_t L_t \tag{43}$$

Note: notation is different from above!! $n_t - 1$ is here net growth rate of L_t

y_t = period-t income per adult

Y_t= total output

Production function: land, labor as inputs

$$Y_t = F(X, L_t) \tag{44}$$

X=land (here constant)

Adult's budget constraint:

$$c_t = y_t - qn_t \tag{45}$$

Utility

$$U_t = U(c_t, n_t) \tag{46}$$

Parametric example: logarithmic utility, Cobb-Douglas production

$$U_t = (1 - \beta)\ln(c_t) + \beta \ln(n_t) \tag{47}$$

Utility maximization gives

$$n_t = \left(\frac{\beta}{q}\right) y_t \tag{48}$$

$$\begin{aligned} Y_t &= Z(X)^{\alpha}(L_t)^{1-\alpha} \\ y_t &= \frac{Y_t}{L_t} = Z\left(\frac{X}{L_t}\right)^{\alpha} \end{aligned} \tag{49}$$

Dynamics equation for (adult) population:

$$L_{t+1} = n_t L_t = \left(\frac{\beta}{q}\right) Z \left(\frac{X}{L_t}\right)^{\alpha} L_t = \left(\frac{\beta}{q}\right) Z X^{\alpha} L_t^{1-\alpha} \tag{50}$$

Illustrate dynamics, steady state in 45°-diagram

Steady state population:

$$\overline{L} = \left[\left(\frac{\beta}{q} \right) Z X^{\alpha} \right]^{\frac{1}{\alpha}} = \left[\left(\frac{\beta}{q} \right) Z \right]^{\frac{1}{\alpha}} X \tag{51}$$

Steady state output per worker:

$$\overline{y} = \frac{q}{\beta} \tag{52}$$

What is the effect on output per worker from rise in productivity, Z? Steady state and transition? (Let $Z = Z^*$ for $t < \widehat{t}$ and $Z = Z^{**} > Z^*$ for $t \geq \widehat{t}$.)

Taking stock

Different models, different assumptions, different results

Some questions of interest to development economists:

- Should we expect economies to converge over time to the same steady state? Role of policy? Changing parameters? Productivity?

- One or many steady states? Poverty traps? Value of "big push" policies? Foreign aid?

- Should we expect improved productivity to even affect living standards in steady state? (Not in a Malthusian model.) If not, is it worthwhile to even try to raise productivity?