Econ 5700 slides

Growth models: Solow, Diamond, Malthus

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Basics: terminology etc.

Growth models are dynamic models: involve time, t

Here focus on *discrete-time* models: the variable t (time) is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Variables usually written x_t

Not "$x(t)$" which is typically continuous time notation

Discrete-time variable cannot be differentiated with respect to t

Dynamics described by difference equations: $x_{t+1} = \phi(x_t)$

Difference equations

Mostly on the form $x_{t+1} = \phi(x_t)$

$\overline{x}$ is a steady state (or steady-state equilibrium) to $x_{t+1} = \phi(x_t)$ if, and only if, $\overline{x} = \phi(\overline{x})$

Means that $\overline{x}$ is a *fixed point* to $\phi(x)$

Stability

$\overline{x}$ is locally stable if, and only if, $\phi'(\overline{x}) \in (-1, 1)$

$\overline{x}$ is globally stable if x_t converges to $\overline{x}$ regardless of starting value, x_0

- Usually equivalent to $\overline{x}$ being (a) locally stable, and (b) unique

If $\phi'(\overline{x}) < 0$ the steady state is called *oscillatory*

Locally stable steady state: $\phi'(\overline{x}) \in (0, 1)$

Locally unstable steady state: $\phi'(\overline{x}) > 1$

Oscillatory locally stable steady state: $\phi'(\overline{x}) \in (-1, 0)$

Oscillatory locally non-stable steady state: $\phi'(\overline{x}) < -1$

Production functions

Standard setting: Y=output, K (capital) and L (labor)

K, L referred to as inputs

$$Y = F(K, L) \tag{1}$$

Usually assumed to satisfy:

(1) Positive marginal products:

$$F_K(\cdot) > 0, F_L(\cdot) > 0 \tag{2}$$

(2) Diminishing marginal products:

$$F_{KK}(\cdot) < 0, F_{LL}(\cdot) < 0 \tag{3}$$

(3) The Inada condition:

$$\begin{aligned} \lim_{Z\to 0} F_Z(\cdot) &= \infty \\ \lim_{Z\to\infty} F_Z(\cdot) &= 0 \end{aligned} \tag{4}$$

for $Z = K, L$

(4) Constant Returns to Scale (CRS):

$$\lambda F(K, L) = F(\lambda K, \lambda L) \tag{5}$$

for all $\lambda > 0$

Intensive-form production function

Let lower-case variables denote per-worker levels

CRS implies

$$y = \frac{Y}{L} = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = F(k, 1) \equiv f(k) \tag{6}$$

Assumptions (1)-(3) imply:

$$\begin{aligned} f'(k) &> 0 \\ f''(k) &< 0 \\ \lim_{k \to 0} f'(k) &= \infty \\ \lim_{k \to \infty} f'(k) &= 0 \end{aligned} \tag{7}$$

Also, using l'Hôpital's Rule:

$$\lim_{k \to 0} \frac{f(k)}{k} = \lim_{k \to 0} \frac{f'(k)}{1} = \infty \tag{8}$$

Factor prices

Atomistic firms take factor prices as given when maximizing profits:

$$\max_{K,L} F(K,L) - \delta K - wL - rK \tag{9}$$

w=wage rate, r=real interest rate, δ=depreciation rate

$F(K,L) - \delta K$=net output (output net of capital depreciation)

Rewrite $F(K,L) = Lf(k) = Lf(\frac{K}{L})$

Exercise: show that

$$\begin{aligned} w &= F_L(K,L) = f(k) - f'(k)k \\ r &= F_K(K,L) - \delta = f'(k) - \delta \end{aligned} \tag{10}$$

Parametric examples of production functions

Cobb-Douglas:

$$F(K,L) = K^{\alpha}L^{1-\alpha} \tag{11}$$
$$f(k) = k^{\alpha}$$

CES (various formulations):

$$F(K,L) = [\alpha K^{\sigma} + (1-\alpha)L^{\sigma}]^{\frac{1}{\sigma}} \tag{12}$$
$$f(k) = [\alpha k^{\sigma} + (1-\alpha)]^{\frac{1}{\sigma}}$$

where $\sigma \in (-\infty, 1]$, $\sigma \neq 0$

Note: CES does not always satisfy Inada

The Solow Growth Model

Discrete time setting: the time variable t is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Notation:

$K_t =$ capital in period t

$\delta =$ depreciation rate

$s =$ rate of saving/investment out of income, Y_t

Evolution of capital stock:

$$K_{t+1} = sY_t + (1 - \delta)K_t \tag{13}$$

L_t = population/labor force in period t

n = (net) growth rate of population

$$L_{t+1} = (1+n)L_t \tag{14}$$

Assume $n > 0$, $\delta \in (0,1]$; $s \in (0,1]$

$$Y_t = F(K_t, L_t) \tag{15}$$

Task: find difference equation for $k_t = K_t/L_t$, on the form: $k_{t+1} = \phi(k_t)$

Use (13) to (15), and $y_t = Y_t/L_t = F(k_t, 1) = f(k_t)$, to get

$$\begin{aligned} \frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) \\ = \frac{sY_t + (1-\delta)K_t}{L_t} = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t \end{aligned} \tag{16}$$

Or:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{17}$$

Properties of $\phi(k_t)$

$$
\begin{aligned}
\phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} f'(k_t) > 0 \\
\phi''(k_t) &= \tfrac{s}{1+n} f''(k_t) < 0 \\
\lim_{k_t \to 0} \phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to 0} f'(k_t) = \infty \\
\lim_{k_t \to \infty} \phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to \infty} f'(k_t) = \tfrac{1-\delta}{1+n} < 1
\end{aligned}
\tag{18}
$$

Together these guarantee: existence, uniqueness, and stability of steady state

(Uniqueness except for the trivial one where $k_t = 0$)

Illustrate in 45°-diagram

Check: If one of the 4 properties above change, how come existence, uniqueness, and stability are no longer guaranteed?

Problems: relaxing assumption (3); and (b) letting saving rate (s) differ across factor payments

Parametric example: Cobb-Douglas production

$$
\begin{aligned}
Y_t &= Z K_t^{\alpha} L_t^{1-\alpha} \\
y_t &= Z k_t^{\alpha}
\end{aligned}
\tag{19}
$$

$$
k_{t+1} = \frac{sZk_t^{\alpha} + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{20}
$$

Steady state capital stock per worker:

$$
\overline{k} = \left(\frac{sZ}{n+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{21}
$$

Steady state output per worker:

$$
\overline{y} = Z^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{22}
$$

Illustrate dynamics, steady state in 45°-diagram

What is effect on output per worker from rise in productivity, Z? Steady state and transition?

The Diamond Overlapping Generations Model

Agents live in two periods: working age, retirement

L_t = number young (working) agents in period t; $L_{t+1} = (1+n)L_t$

$c_{1,t}$ = consumption of working agent in period t

$c_{2,t}$ = consumption of retired agent in period t

s_t = saving of working agent in period t

$R_{t+1} = 1 + r_{t+1}$ = gross interest rate on savings held from period t to $t+1$

w_t = period-t wage rate

Consider agent young/working in period t

Budget constraints

$$c_{1,t} = w_t - s_t \tag{23}$$

$$c_{2,t+1} = R_{t+1} s_t \tag{24}$$

Utility:

$$U_t = U(c_{1,t}, c_{2,t+1}) \tag{25}$$

Optimal savings decision given by $s(w_t, R_{t+1})$, defined from

$$s(w_t, R_{t+1}) = \underset{s_t \in [0, w_t]}{\arg\max} U(w_t - s_t, R_{t+1} s_t) \tag{26}$$

Capital accumulation:

Total savings in period t = total capital stock in period $t+1$

$$s(w_t, R_{t+1})L_t = K_{t+1} \tag{27}$$

Recall: $L_{t+1} = (1+n)L_t$

$$\frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) = s(w_t, R_{t+1}) \tag{28}$$

Factor prices (recall) given by marginal products

$$w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t) \tag{29}$$

$$R_{t+1} = f'(k_{t+1}) + 1 - \delta \equiv R(k_{t+1}) \tag{30}$$

Thus:

$$k_{t+1} = \phi(k_t) \tag{31}$$

where $\phi(k_t)$ is defined from

$$\phi(k_t) = \frac{s\{w(k_t), R(\phi(k_t))\}}{1+n} \tag{32}$$

The very general case: possibility of indeterminacies

At given k_t, k_{t+1} is not necessarily unique. To see this, hold k_t constant, and replace $\phi(k_t)$ in (32) with k_{t+1}, i.e., $k_{t+1} = s\{w(k_t), R(k_{t+1})\}/(1+n)$.

Always true that $R'(k_{t+1}) < 0$. Assume $\partial s(\cdot)/\partial R_{t+1} < 0$ (over some interval). Then the right-hand side is increasing in k_{t+1}. Since the left-hand side is too, there could be multiple k_{t+1} for given k_t

Intuition: if savings are high, interest rates are low (=marginal product of capital), sustaining high savings [given $\partial s(\cdot)/\partial R_{t+1} < 0$]

Called an *indeterminacy;* implies backward bending $\phi(k_t)$

Illustration: see 45°-diagram

Parametric example: logarithmic utility, Cobb-Douglas production

Log utility:

$$U_t = (1 - \beta)\ln(c_{1,t}) + \beta\ln(c_{2,t+1}) \tag{33}$$

First-order condition:

$$-(1 - \beta)(w_t - s_t)^{-1} + \beta s_t^{-1} = 0 \tag{34}$$

Solving for s_t:

$$s_t = \beta w_t \tag{35}$$

Cobb-Douglas production:

$$\begin{aligned} Y_t &= Z K_t^{\alpha} L_t^{1-\alpha} \\ y_t &= Z k_t^{\alpha} \\ w_t &= (1 - \alpha) Z k_t^{\alpha} \end{aligned} \tag{36}$$

$$k_{t+1} = \frac{\beta(1-\alpha)Zk_t^{\alpha}}{1+n} = \phi(k_t) \qquad (37)$$

Steady state capital stock per worker:

$$\overline{k} = \left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{1}{1-\alpha}} \qquad (38)$$

Steady state output per worker:

$$\overline{y} = Z\left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{\alpha}{1-\alpha}} = Z^{\frac{1}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+n}\right]^{\frac{\alpha}{1-\alpha}} \qquad (39)$$

Illustrate dynamics, steady state in 45°-diagram; no indeterminacies

What is the effect on output per worker from rise in productivity, Z? Steady state and transition?

Poverty traps in the Diamond OLG model

Keep assumption about logarithmic utility ($s_t = \beta w_t$) but allow for richer production function

$$k_{t+1} = \frac{\beta \left[f(k_t) - f'(k_t) k_t \right]}{1+n} \equiv \phi(k_t) \tag{40}$$

$$\phi'(k_t) = \frac{-\beta}{1+n} f''(k_t) k_t > 0 \tag{41}$$

$$\phi''(k_t) = \frac{-\beta}{1+n} \left[f'''(k_t) k_t + f''(k_t) \right] \gtrless 0 \tag{42}$$

The sign of $\phi''(k_t)$ depends on the *third derivative* of production function, about which we have not made any assumptions

Multiple steady states possible, even with neoclassical production function

Illustrate dynamics, (all) steady states in 45°-diagram

Which are stable, unstable?

The Malthus Model

Agents live in two periods: children, adults

$L_t =$ number adult (working) agents in period t

$c_t =$ consumption of adult agent in period t

$q =$ cost per child (can be interpreted as consumption per child); exogenous

$n_t =$ number of children per adult

$$L_{t+1} = n_t L_t \tag{43}$$

Note: notation is different from above!! $n_t - 1$ is here net growth rate of L_t

y_t = period-t income per adult

Y_t= total output

Production function: land, labor as inputs

$$Y_t = F(X, L_t) \tag{44}$$

X=land (here constant)

Adult's budget constraint:

$$c_t = y_t - qn_t \tag{45}$$

Utility

$$U_t = U(c_t, n_t) \tag{46}$$

Parametric example: logarithmic utility, Cobb-Douglas production

$$U_t = (1 - \beta)\ln(c_t) + \beta \ln(n_t) \tag{47}$$

Utility maximization gives

$$n_t = \left(\frac{\beta}{q}\right) y_t \tag{48}$$

$$\begin{aligned} Y_t &= Z(X)^{\alpha}(L_t)^{1-\alpha} \\ y_t &= \frac{Y_t}{L_t} = Z\left(\frac{X}{L_t}\right)^{\alpha} \end{aligned} \tag{49}$$

Dynamics equation for (adult) population:

$$L_{t+1} = n_t L_t = \left(\frac{\beta}{q}\right) Z \left(\frac{X}{L_t}\right)^{\alpha} L_t = \left(\frac{\beta}{q}\right) Z X^{\alpha} L_t^{1-\alpha} \tag{50}$$

Illustrate dynamics, steady state in 45°-diagram

Steady state population:

$$\overline{L} = \left[\left(\frac{\beta}{q} \right) Z X^{\alpha} \right]^{\frac{1}{\alpha}} = \left[\left(\frac{\beta}{q} \right) Z \right]^{\frac{1}{\alpha}} X \tag{51}$$

Steady state output per worker:

$$\overline{y} = \frac{q}{\beta} \tag{52}$$

What is the effect on output per worker from rise in productivity, Z? Steady state and transition?

Taking stock

Different models, different assumptions, different results

Some questions of interest to development economists:

- Should we expect economies to converge over time to the same steady state? Role of policy? Changing parameters? Productivity?

- One or many steady states? Poverty traps? Value of "big push" policies? Foreign aid?

- Should we expect improved productivity to even affect living standards in steady state? (Not in a Malthusian model). If not, is it worthwhile to even try to raise productivity?