Lecture Notes in Growth Theory – Part IV

More on growth in the very long run

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Unified frameworks

Stylized facts about long-run development in Western Europe: 3 stages passed on way to the Industrial Revolution

1. *Malthusian Regime* (1000's of years B.C. to 1500's): slowly growing population and per-capita income levels; positive relationship between the two

2. *Post-Malthusian Regime* (1500's to mid 1800's): faster growth in population and per-capita income; still positive relationship between the two

3. *Modern Growth Regime* (mid 1800's till today); lower population growth rate but accelerated growth in per-capita income; negative relationship between the two

Galor and Weil (2000): *unified framework*

Means: the model should explain full transition – all of the Three Regimes – *endogenously*

Contrast to the story in Becker-Murphy-Tamura (1990), Barro-Becker (1989) ("Old school"); there we had:

- One steady-state with high fertility (population growth), non-growing per-capita income
- Another rich steady state (or balanced growth path) with low fertility
- Shocks can make the economy jump from one to the other

Shortcomings:

- Not all 3 regimes explained; only Post-Malthusian and Modern Growth Regime
- Transition from one regime to the next not explained

The Galor-Weil model: many interacting mechanisms, seeks to explain "everything:"
endogenous technological progress
endogenous fertility
endogenous education choice, human capital
land in fixed supply

X = land; A_t = technology (land augmenting); XA_t = effective resources

Effective resources per worker: $x_t = (A_t X)/L_t$

h_t = human capital per worker

Income per worker:

$$z_t = h_t^{\alpha} x_t^{1-\alpha} \tag{1}$$

g_{t+1} = technological progress from period t to $t+1$

$$g_{t+1} = \frac{A_{t+1} - A_t}{A_t} \tag{2}$$

Human capital production

$$h_{t+1} = h(e_{t+1}, g_{t+1}) \tag{3}$$

e_{t+1} = education invested in kids in period t

Assumptions:

$$\begin{array}{ll} h_e(e,g) > 0 & h_{ee}(e,g) < 0 \\ h_g(e,g) < 0 & h_{gg}(e,g) > 0 \end{array} \tag{4}$$

Interpretation:
Education raises human capital; declining marginal effect
Technological progress *reduces* human capital (making knowledge obsolete); an "erosion effect," declining on the margin

Also assume:

$$h_{eg}(e,g) > 0 \tag{5}$$

Interpretation: technological progress raises the *return* to investing in education; erosion effect declines in eduction

Utility:

$$u_t = (1-\gamma)\ln c_t + \gamma \ln(n_t h_{t+1}) \tag{6}$$

Budget constraint:

$$c_t = z_t \left[1 - (\tau + e_{t+1}) n_t\right] \tag{7}$$

Each unit of education costs one unit of time; τ is a fixed time cost; so each child costs $(\tau + e_{t+1})$ units of time to rear

(More general formulation; see paper: let τ^e = time cost per unit of education; τ^q = fixed time cost per child; then total time cost per child $= \tau^q + \tau^e e_{t+1}$)

Maximize utility in (6) subject to four constraints:
Budget constraint in (7)
Human capital production function in (3),
Subsistence consumption constraint: $c_t \geq \widetilde{c}$; and
Non-negative constraint on education: $e_{t+1} \geq 0$

FOC's depend on whether $c_t \geq \widetilde{c}$ and $e_{t+1} \geq 0$ are binding

FOC for n_t implies:

$$n_t[\tau + e_{t+1}] = \begin{cases} \gamma & \text{if } z_t \geq \widetilde{z} \\ 1 - \frac{\widetilde{c}}{z_t} & \text{if } z_t \leq \widetilde{z} \end{cases} \qquad (8)$$

where $\widetilde{z} = \widetilde{c}/(1-\gamma)$; $z_t \leq \widetilde{z} \Leftrightarrow c_t \geq \widetilde{c}$ binding

Total time spent on children is rising in potential income, z_t, up until $z_t = \widetilde{z}$; then constant; see indifference curve diagram in Galor and Weil (2000)

Education

Optimal e_{t+1} is given by:

$$G(e_{t+1}, g_{t+1}) \begin{cases} = 0 & \text{if } e_{t+1} > 0 \\ > 0 & \text{if } e_{t+1} = 0 \end{cases} \qquad (9)$$

where

$$G(e_{t+1}, g_{t+1}) = (\tau + e_{t+1})\, h_e(e_{t+1}, g_{t+1}) - h(e_{t+1}, g_{t+1}) \qquad (10)$$

Assumptions about $h(e_{t+1}, g_{t+1})$ made above imply:

$$G_e(e_{t+1}, g_{t+1}) = (\tau + e_{t+1}) \underbrace{h_{ee}(e_{t+1}, g_{t+1})}_{<0} < 0 \tag{11}$$

$$G_g(e_{t+1}, g_{t+1}) = (\tau + e_{t+1}) \underbrace{h_{eg}(e_{t+1}, g_{t+1})}_{>0} - \underbrace{h_g(e_{t+1}, g_{t+1})}_{<0} > 0 \tag{12}$$

Use Implicit Function theorem to see that e_{t+1} is increasing in g_{t+1}:

$$e'(g_{t+1}) = -\frac{G_g(e_{t+1}, g_{t+1})}{G_e(e_{t+1}, g_{t+1})} > 0 \tag{13}$$

if $e_{t+1} > 0$

Next assume:

$$G(0,0) = \tau h_e(0,0) - h(0,0) < 0 \tag{14}$$

Implies that there exists some $\widehat{g} > 0$, such that e_{t+1} is constrained to zero if $g_{t+1} < \widehat{g}$

Thus:

$$e(g_{t+1}) \begin{cases} > 0 & \text{if } g_{t+1} > \widehat{g} \\ = 0 & \text{if } g_{t+1} \leq \widehat{g} \end{cases} \tag{15}$$

Technological progress

Assumed to depend on education and population size

$$g_{t+1} = g(e_t; L) \tag{16}$$

Assume that

$$g(0; L) > 0 \tag{17}$$

Some technological progress also in absence of education

Dynamics for e_t and g_t

First: dynamic analysis done holding L constant
Later: "tie it all together" by letting population expand over time, linking it to the endogenous fertility rate, n_t
Constant L is good approximation if n_t close to one, and population growth close to zero

Dynamical system for e_t and g_t

$$\begin{aligned} e_{t+1} &= e(g(e_t; L)) \\ g_{t+1} &= g(e_t; L) \end{aligned} \tag{18}$$

Three types of dynamic configuration possible, depending on population size

1. Small population, L^{low}: no education, slow technological progress

2. Moderate population, L^{m}: multiple steady states

3. Large population, L^{high}: unique steady state with fast technological progress and high education

Scenario: slowly expanding population (from L^{low} to L^{high})

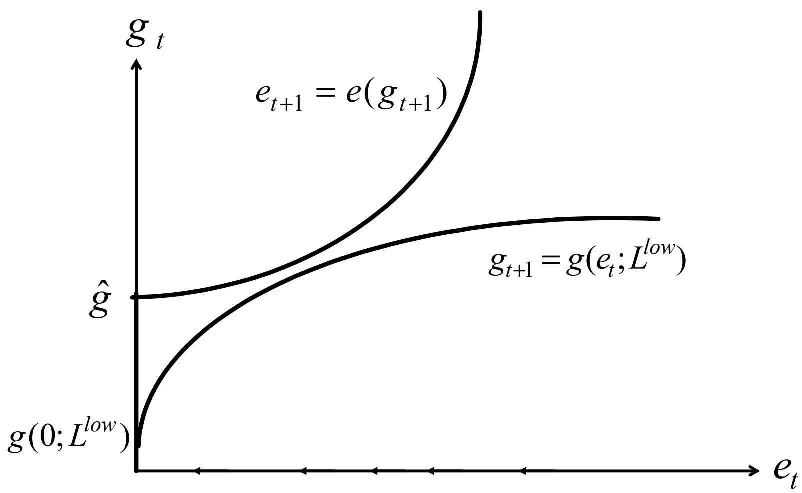
Initially nothing happens to education; $e = 0$ in steady state

But slowly rising rates of technological progress, as population expands

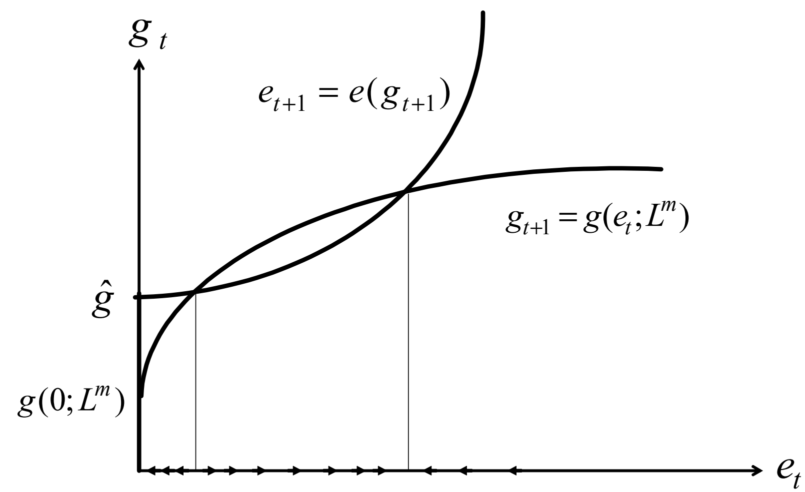
As $g(0; L) > \widehat{g}$ the whole configuration changes: spurt in technological progress and rise in education time

Technological change and levels of education rise; jointly reinforcing one another; rising technological progress not driven by expanding population any longer

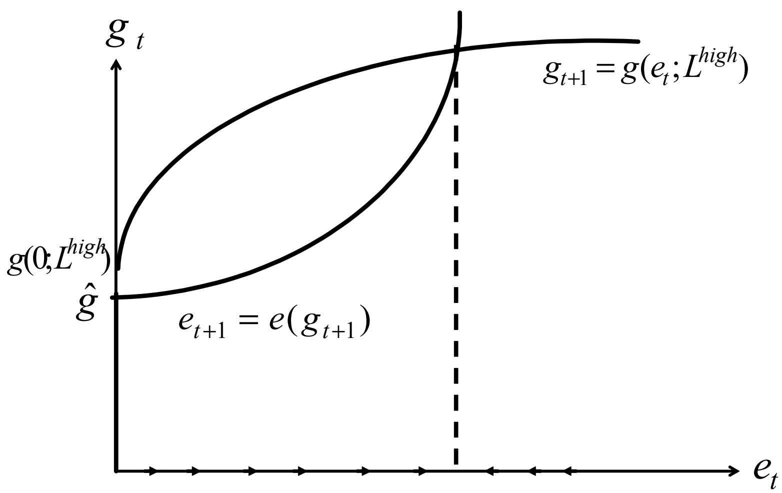
Expanding population like ticking time bomb: once it reaches a threshold everything happens at once



g_t
$e_{t+1} = e(g_{t+1})$
$g_{t+1} = g(e_t; L^{low})$
$\hat{g}$
$g(0; L^{low})$
e_t



g_t
$e_{t+1} = e(g_{t+1})$
$g_{t+1} = g(e_t; L^m)$
$\hat{g}$
$g(0; L^m)$
e_t



Parametric example

Let

$$h_{t+1} = h(e_{t+1}, g_{t+1}) = \frac{e_{t+1} + \rho\tau}{e_{t+1} + \rho\tau + g_{t+1}}, \quad (19)$$

where $\rho \in (0, 1)$

Let

$$g_{t+1} = g(e_t; L) = (e_t + \rho\tau)a(L) \quad (20)$$

where, $a(0) > 0$, $a'(L) > 0$ and $\lim_{L\to\infty} a(L) \equiv a^* \in (0, \infty)$

Interpretation: the fixed time cost of rearing children, τ, builds human capital to some extent but not as effectively as education, e_{t+1}; thus $\rho < 1$

Optimal education time becomes

$$e(g_{t+1}) = \max\left\{0, \{g_{t+1}\tau(1-\rho)\}^{1/2} - \rho\tau\right\} \quad (21)$$

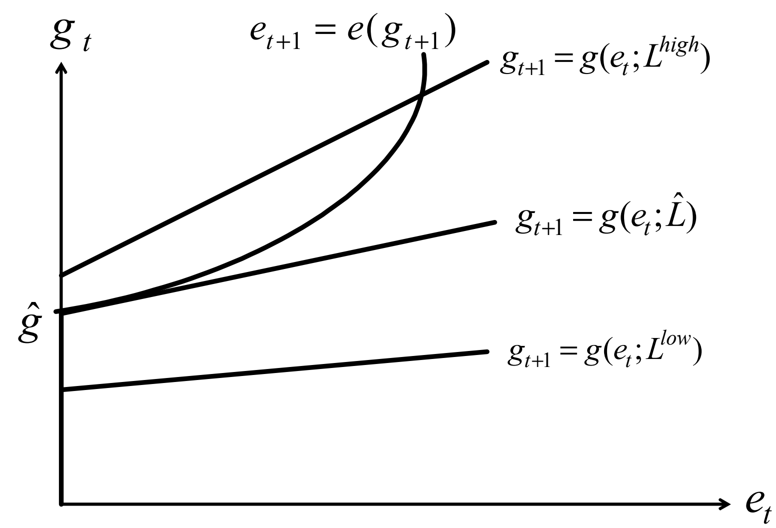
The threshold level of technological change above which education time is operative (not constrained to zero):

$$\hat{g} = \frac{\rho^2 \tau}{1 - \rho} \qquad (22)$$

And the associated level of population, denoted $\widehat{L}$, is given by:

$$a(\widehat{L}) = \frac{\rho}{1 - \rho} \qquad (23)$$

Illustration: note that $g(e_t; L)$ now linear in e_t; $a(L)$ determines both the slope and the intercept



$L^{\text{low}} < \widehat{L} < L^{\text{high}}$

Two configurations possible

1. $L \leq \widehat{L}$; and thus $g(0; L) = \rho\tau a(L) < \widehat{g}$; only steady state that exists is one where technological progress slow, and parents do not invest in education:
$$\begin{aligned} e^0(L) &= 0 \\ g^0(L) &= \rho\tau a(L) \end{aligned} \tag{24}$$

2. $L \geq \widehat{L}$; only steady state that exists is one where technological progress is rapid, and parents invest in education:
$$\begin{aligned} \overline{e}(L) &= \tau\left[(1-\rho)a(L) - \rho\right] \\ \overline{g}(L) &= \tau(1-\rho)[a(L)]^2 \end{aligned} \tag{25}$$

Note that $e^0(\widehat{L}) = \overline{e}(\widehat{L})$ and $g^0(\widehat{L}) = \overline{g}(\widehat{L})$

Can also be seen that human capital is the same in the two steady states:
$$\begin{aligned} h\left(e^0(L), g^0(L)\right) &= h\left(\overline{e}(L), \overline{g}(L)\right) \\ &= \tfrac{1}{1+a(L)} \equiv h(L) \end{aligned} \tag{26}$$

where we note that $h'(L) < 0$, since $a'(L) > 0$

Dynamics of L_t and A_t

Dynamical system approximated around either one of the above steady states for e and g

Difference equation for A_t
$$A_{t+1} = \begin{cases} \left[1 + g^0(L_t)\right] A_t & \text{if } L_t \leq \widehat{L} \\ \left[1 + \overline{g}(L_t)\right] A_t & \text{if } L_t \geq \widehat{L} \end{cases} \tag{27}$$

To find difference equation for L_t we need the fertility rate

Use optimal fertility in (8) and optimal education in (21), to write fertility as function of g_{t+1} and potential income, z_t

Four cases:

I. $L_t \leq \widehat{L}$ and $z_t \geq \widetilde{z}$: education time constrained to zero, but consumption not constrained to subsistence

$$n_t = \frac{\gamma}{\tau} > 1$$

(assuming $\gamma > \tau$). That is: fertility is constant and independent of both g_{t+1} and z_t

II. $L_t \leq \widehat{L}$ and $z_t \leq \widetilde{z}$: education time constrained to zero, and consumption constrained to subsistence

$$n_t = \frac{1 - \frac{\widetilde{c}}{z_t}}{\tau}$$

That is: fertility is independent of g_{t+1} but increasing in z_t

III. $L_t \geq \widehat{L}$ and $z_t \leq \widetilde{z}$: education time not constrained to zero, but consumption constrained to subsistence

$$n_t = \frac{1 - \frac{\widetilde{c}}{z_t}}{\tau + e(g_{t+1})}$$

That is: fertility is falling in g_{t+1} and increasing in z_t

IV. $L_t \geq \widehat{L}$ and $z_t \geq \widetilde{z}$: education time not constrained to zero, and consumption not constrained to subsistence

$$n_t = \frac{\gamma}{\tau + e(g_{t+1})}$$

That is: fertility is falling in g_{t+1} and independent of z_t

Next: find expressions for $e(g_{t+1})$ and z_t in terms of L_t and A_t

Education, $e(g_{t+1})$, in steady state associated with $L_t \geq \widehat{L}$: function of L_t; see (25)

$$e(g_{t+1}) = \overline{e}(L_t) = \tau\left[(1-\rho)a(L_t) - \rho\right] \qquad (28)$$

Income, z_t, given by: $z_t = h_t^{\alpha} x_t^{1-\alpha}$

Substitute $h(L_t)$ for h_t, where (recall) $h'(L_t) < 0$; see (26)

Recall: $x_t = (A_t X)/L_t$

This gives:

$$z_t = [h(L_t)]^{\alpha}\left[\frac{A_t X}{L_t}\right]^{1-\alpha} \equiv z(L_t, A_t) \qquad (29)$$

The four cases again:

I. $L_t \leq \widehat{L}$ and $z(L_t, A_t) \geq \tilde{z}$:

$$n_t = \frac{\gamma}{\tau} > 1$$

II. $L_t \leq \widehat{L}$ and $z(L_t, A_t) \leq \tilde{z}$:

$$n_t = \frac{1 - \frac{\widetilde{c}}{z(L_t, A_t)}}{\tau}$$

III. if $L_t \geq \widehat{L}$ and $z(L_t, A_t) \leq \tilde{z}$:

$$n_t = \frac{1 - \frac{\widetilde{c}}{z_t}}{\tau + \overline{e}(L_t)} = \frac{1 - \frac{\widetilde{c}}{z(L_t, A_t)}}{\tau\left[(1-\rho)[1 + a(L_t)\right]}$$

IV. $L_t \geq \widehat{L}$ and $z(L_t, A_t) \geq \tilde{z}$:

$$n_t = \frac{\gamma}{\tau + \overline{e}(L_t)} = \frac{\gamma}{\tau\left[(1-\rho)[1 + a(L_t)\right]}$$

Difference equation for L_t:

$$L_{t+1} = \begin{cases} \frac{\gamma L_t}{\tau} & \text{if } L_t \leq \widehat{L} \text{ and } z(L_t, A_t) \geq \widetilde{z} \\ \left\{ \frac{1 - \frac{\widetilde{c}}{z(L_t, A_t)}}{\tau} \right\} L_t & \text{if } L_t \leq \widehat{L} \text{ and } z(L_t, A_t) \leq \widetilde{z} \\ \left\{ \frac{1 - \frac{\widetilde{c}}{z(L_t, A_t)}}{\tau[(1-\rho)[1+a(L_t)]} \right\} L_t & \text{if } L_t \geq \widehat{L} \text{ and } z(L_t, A_t) \leq \widetilde{z} \\ \left\{ \frac{\gamma}{\tau[(1-\rho)[1+a(L_t)]} \right\} L_t & \text{if } L_t \geq \widehat{L} \text{ and } z(L_t, A_t) \geq \widetilde{z} \end{cases} \quad (30)$$

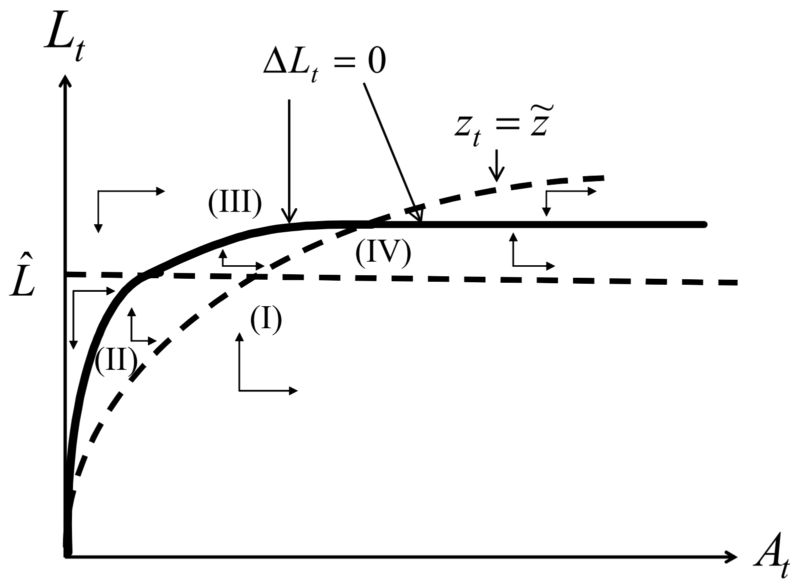
Together (27) and (30) constitute a dynamical system for A_t and L_t

Phase diagram; vertical axis: L_t, horizontal axis: A_t

A_t always growing (i.e., $A_{t+1} > A_t$) since g_{t+1} always positive; no $(\Delta A_t = 0)$-locus
However: A_t grows *faster* north of $\widehat{L}$

Locus along which $\Delta L_t = 0$ given by $L_{t+1} = L_t$ $(n_t = 1)$
differs across regions; see phase diagram

Start in region (II): slow growth in population and technology; path close to $(\Delta L_t = 0)$-locus
Enter region (III): faster growth in technology and income; population growth faster too, due to income effect (subsistence constraint still binding)
Enter region (IV): continued fast growth in technology but slowdown in population growth as subsistence constraint no longer binding



Epidemics

Another model replicating Galor and Weil's three regimes: Lagerlöf (2003)

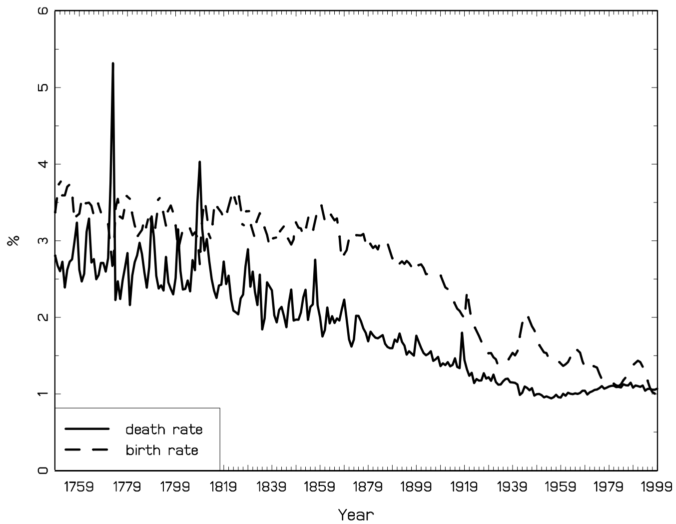
Story:

Population hit by shocks to mortality, epidemics
Industrial revolution result of series of mild shocks, causing population expansion
Population expansion causes rise in the return to educating kids (scale effect)
At some point a non-negativity constraint on education time stops to be binding; parents substitute from quantity to quality

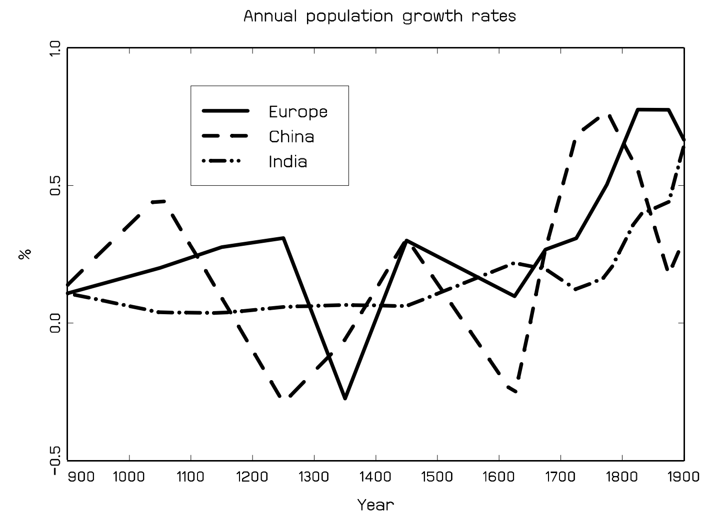
Framework similar to Becker, Murphy, and Tamura (1990)

Differences to Galor and Weil:
Explains stochastic nature of mortality and why volatility in mortality declined
Generates time path, easier to see the three regimes

Demographic transition in Sweden



Fluctuations in population growth in world regions



Consumption and production

$$C_t = Dl_t(L + H_t) \tag{31}$$

C_t = output = consumption, D = productivity parameter, l_t = time input in goods production

$L + H_t$ = time-augmenting human capital

L from "nature", H_t from parents

Time

$$1 = l_t + (v + h_t)B_t \tag{32}$$

B_t = number of born children

$v + h_t$ = time spent on each born child; adult time endowment = 1

v = fixed time cost of rearing one child

h_t = time spent educating each child

Mortality

$$T_t = T\left(H_t/P_t, \omega_t\right) = \frac{H_t/P_t}{\omega_t + H_t/P_t} \tag{33}$$

T_t = fraction of B_t born children who survive to adulthood.

P_t = adult population in period t

ω_t = epidemic shock > 0; e.g. $\ln \omega_t \sim N(\mu, \sigma)$

Why this form? $T_t = \frac{H_t/P_t}{\omega_t + H_t/P_t}$

Mortality rate between zero and one

Epidemic shock raises mortality (lowers the survival rate T_t)

Lots of human capital and/or a low population $\Rightarrow$ low mortality

If human capital grows at a faster rate than population $\Rightarrow H_t/P_t$ approaches infinity $\Rightarrow$ mortality approaches zero, epidemics have no effect

Human capital

$$H_{t+1} = A(P_t)\,[L + H_t]\,(\rho v + h_t) \tag{34}$$

$A(P_t)$ = productivity in human capital production, "scale effect"

Positive effect on learning in regions with shorter geographical distance between people (*cities)*; consistent with empirical evidence

ρv = the direct inheritance of human capital, $\rho \in (0, 1)$, drives the dynamics of human capital at early stages of economic development, when $h_t = 0$

For calibration, we use this functional form:

$$A(P_t) = A^* - \widetilde{A} + \widetilde{A}\left(\frac{P_t}{\eta + P_t}\right) = A^* - \widetilde{A}\left(\frac{\eta}{\eta + P_t}\right) \tag{35}$$

$A^* > \widetilde{A}$, $\eta > 0$

Preferences

$$U_t = \ln(C_t) + \alpha \ln(B_t T_t) + \alpha\delta \ln(L + H_{t+1}) \tag{36}$$

Assume $\delta \in (\rho, 1)$ to guarantee the existence of an interior solution (see soon)

Max subject to expressions for H_{t+1} and C_t

$$\max_{(h_t,B_t)\in\Re^2_+} \ln[1-(v+h_t)B_t](L+H_t)]+$$

$$\alpha \ln(B_t T_t) + \alpha\delta \ln\left\{L + A(P_t)\left[L+H_t\right](\rho v + h_t)\right\} \tag{37}$$

First-order condition for B_t gives

$$B_t = \left(\frac{\alpha}{1+\alpha}\right)\frac{1}{v+h_t} \tag{38}$$

Time spent on children, $(v+h_t)B_t =$ constant fraction of the unit time endowment, following from log utility

First-order condition for h_t complicated

Trick: substitute optimal B_t in (38) and expression for H_{t+1} in (34) into U_t

FOC for h_t gives:

$$h_t = \frac{1}{1-\delta}\left[v(\delta-\rho) - \frac{L}{A(P_t)(L+H_t)}\right] \tag{39}$$

If RHS<0, $h_t = 0$

h_t operative (i.e., not constrained to zero) for high enough $A(P_t)(L+H_t)$

Use expression for $A(P_t)$ in (35)

Define

$$\Gamma(H_t) = \eta\left(\frac{\widetilde{A}}{A^* - \frac{L}{v(\delta-\rho)[L+H_t]}} - 1\right) \tag{40}$$

Then $h_t > 0$ if $P_t > \Gamma(H_t)$; else $h_t = 0$

Dynamical system

$$P_{t+1} = \begin{cases} \left(\frac{\alpha P_t}{1+\alpha}\right)\left(\frac{(1-\delta)A(P_t)[L+H_t]}{v(1-\rho)A(P_t)[L+H_t]-L}\right)\left(\frac{H_t/P_t}{\omega_t+H_t/P_t}\right) \\ \text{if } P_t > \Gamma(H_t) \\ \frac{\alpha P_t}{(1+\alpha)v}\left(\frac{H_t/P_t}{\omega_t+H_t/P_t}\right) \\ \text{if } P_t \leq \Gamma(H_t) \end{cases} \quad (41)$$

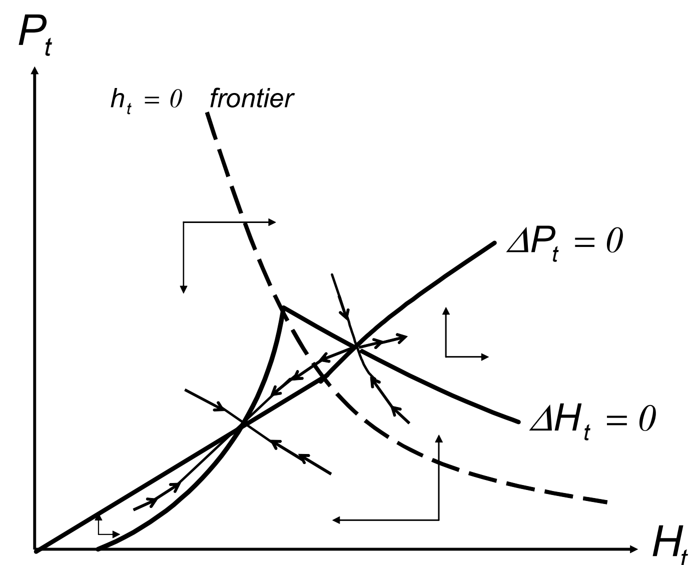
$$H_{t+1} = \begin{cases} \frac{v\delta(1-\rho)A(P_t)[L+H_t]-L}{1-\delta} & \text{if } P_t > \Gamma(H_t) \\ \rho v A(P_t)[L+H_t] & \text{if } P_t \leq \Gamma(H_t) \end{cases} \quad (42)$$

2-dimensional, non-linear, and dependent on epidemic shock, ω_t

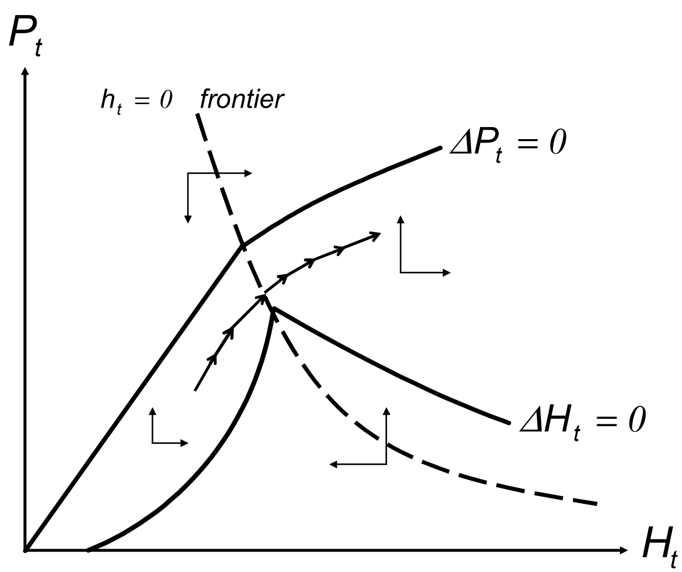
Rig model so that a high-ω economy may be stuck in a locally stable (Malthusian) steady state; low-ω economy converges to a balanced growth path

Illustration: see phase diagrams

High ω



Low ω

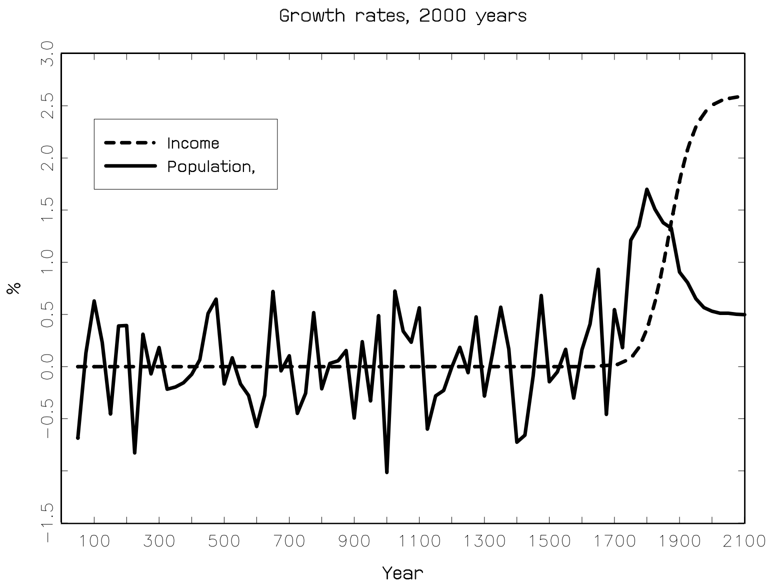


Calibrate and simulate the model:

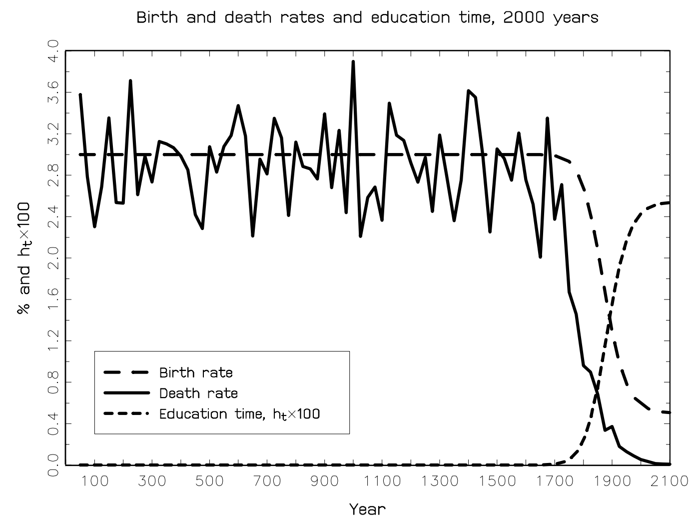
1. choose with H_0 and P_0
2. draw ω_0 from log normal distribution
3. calculate H_1 and P_1
4. draw ω_1
5. calculate H_2 and P_2

....and so on....

Result: see figures



Growth rates, 2000 years
Income
Population,
%
Year



Birth and death rates and education time, 2000 years
Birth rate
Death rate
Education time, $h_t \times 100$
% and $h_t \times 100$
Year