Lecture Notes in Growth Theory – Part II

Growth Models with Endogenous Demographics

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Insight from cross-country growth regressions: fertility and population growth highly correlated with per-capita income growth (and levels)

Calls for growth models with endogenous fertility

Early approaches (1980's): endogenous fertility in dynastic (Ramsey) settings; rich and complex models but worth getting acquainted to

Central components:
(a) *quality-quantity trade-off* in children: having more children means less bequest, lower utility, of each child
(b) *time cost* of children: more capital, higher wages induces substitution from quantity to quality

First: Barro and Becker (1989)
(see also Becker and Barro QJE 1988)

Utility

$$U_t = c_t^\sigma + a(n_t) n_t U_{t+1} \tag{1}$$

$n_t = \#$ of children

$c_t =$ consumption per agent; $\sigma \in (0, 1)$

$a'(n_t) < 0$; weight attached to each child's utility, $a(n_t)$, decreasing in n_t

Idea: decreasing marginal utility of children

$$\begin{aligned} \frac{\partial U_t}{\partial n_t} &> 0 \\ \frac{\partial^2 U_t}{\partial n_t^2} &< 0 \end{aligned} \tag{2}$$

Simple parametric example: $a(n_t) = \alpha n_t^{-\varepsilon}$

$$U_t = c_t^\sigma + \alpha n_t^{1-\varepsilon} U_{t+1} \tag{3}$$

$1 - \sigma - \varepsilon > 0$

Population

$N_t = \#$ of adults in period t

$$N_{t+1} = n_t N_t \tag{4}$$

(agents die after adulthood)

Rewriting utility

$V_t = N_t^{1-\varepsilon} U_t$

$C_t = N_t c_t =$ aggregate consumption

$$V_t = N_t^{1-\varepsilon-\sigma} C_t^{\sigma} + \alpha V_{t+1} \tag{5}$$

Let x_t denote capital (K_t) per adult

$$x_t = \frac{K_t}{N_t} \tag{6}$$

Budget constraint

$$c_t = w_t + (1 + r_t)k_t - n_t[\beta_t + x_{t+1}]$$

or:

$$C_t = N_t w_t + (1 + r_t)K_t - N_t n_t[\beta_t + x_{t+1}] \qquad (7)$$

$\beta_t =$ cost rearing each child (explained later)

$x_{t+1} =$ bequest to each child

Optimality condition for c_t (Euler)

$$\frac{\sigma c_t^{\sigma-1}}{\sigma c_{t+1}^{\sigma-1}} = \left(\frac{c_{t+1}}{c_t}\right)^{1-\sigma} = \alpha n_t^{-\varepsilon}(1 + r_{t+1}) \qquad (8)$$

Note: $\varepsilon = 0$ implies $a(n_t) = \alpha n_t^{-\varepsilon} = \alpha$; brings us back to setting with exogenous fertility

Optimality condition for n_t

See problem set

Gives Eq. (9) in Barro and Becker (1989):

$$\frac{C_{t+1}}{N_{t+1}} = c_{t+1} = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right)[\beta_t(1+r_{t+1}) - w_{t+1}] \tag{9}$$

Intuition: cost of N_{t+1} in terms of C_{t+1}:

1. *Bequest loss:* keep period t spending on children, $N_{t+1}[\beta_t + x_{t+1}]$, constant; then an increase in N_{t+1} reduces $K_{t+1} = N_{t+1}x_{t+1}$ by β_t; this is worth $(1 + r_{t+1})$ in period $t + 1$

2. *Salary gain:* from more labor income in period $t + 1$

$$\underbrace{\beta_t(1+r_{t+1})}_{\text{bequest loss}} - \underbrace{w_{t+1}}_{\text{salary gain}}$$

Child rearing cost

Goods cost: a per child

Time cost: b per child; each unit of time is worth w_t

$$\beta_t = a + bw_t \tag{10}$$

Production

$L_t =$ labor supply $=$ # adults times labor supply per adult $= N_t(1 - bn_t)$

Lower-case variables denote per-labor units (not per adult)

$$Y_t = F(K_t, L_t) = L_t f(k_t) \tag{11}$$

Capital-labor ratio:

$$\frac{K_t}{L_t} = \frac{K_t}{N_t(1 - bn_t)} = \frac{x_t}{1 - bn_t}, \tag{12}$$

where (recall) $x_t = K_t/N_t$ is capital per adult

Rewrite:

$$x_t = (1 - bn_t)k_t \tag{13}$$

Factor prices

$$\begin{aligned} w_t &= f(k_t) - f'(k_t)k_t \\ r_t &= f'(k_t) - \delta \end{aligned} \tag{14}$$

Note: $w_t + (1 + r_t)k_t = f(k_t) + (1 - \delta)k_t$

"Full" income per adult (excluding time cost of children)

$$\begin{aligned}
&[N_t w_t + (1+r_t)K_t]/N_t \\
&= w_t + (1+r_t)x_t \\
&= w_t + (1+r_t)(1-bn_t)k_t \\
&= w_t + (1+r_t)k_t - bn_t(1+r_t)k_t \\
&= f(k_t) + (1-\delta)k_t - bn_t(1+r_t)k_t
\end{aligned} \tag{15}$$

Consumption per adult:

$$c_t = w_t + (1+r_t)x_t - n_t[\beta_t + x_{t+1}]$$

$$= f(k_t) + (1-\delta)k_t - bn_t(1+r_t)k_t - n_t[\beta_t + x_{t+1}] \tag{16}$$

Steady state

$$c = f(k) + (1-\delta)k$$

$$-bn(1+r)k - n\left[\beta + \underbrace{x}_{k(1-bn)}\right] \qquad (17)$$

$$= f(k) + (1-\delta)k - bn(1+r)k$$

$$-n\left[\beta + k(1-bn)\right]$$

From optimal fertility in (9):

$$c = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right)\left[\beta(1+r) - w\right] \qquad (18)$$

Together, these give Eq. (22) in Barro and Becker (1989):

$$f(k) + (1-\delta)k = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right)[\beta(1+r) - w] \\ +n[\beta + k] + bnk[(1+r) - n] \tag{19}$$

Defines fertility as a function of r $(= f'(k) - \delta)$; denote it $n_2(r)$
If $b = 0$ (and β thus constant): $\frac{\partial n_2(r)}{\partial r} < 0$ (or so claims Barro and Becker)
If $b > 0$ (β increasing in k and falling in r): $\frac{\partial n_2(r)}{\partial r} > 0$ possible

Next use steady-state Euler Equation in (8):

$$(c_{t+1}/c_t)^{1-\sigma} = 1 = \alpha n^{-\varepsilon}(1+r) \tag{20}$$

Defines fertility as an increasing function of r

$$n = (\alpha[1+r])^{\frac{1}{\varepsilon}} \equiv n_1(r) \tag{21}$$

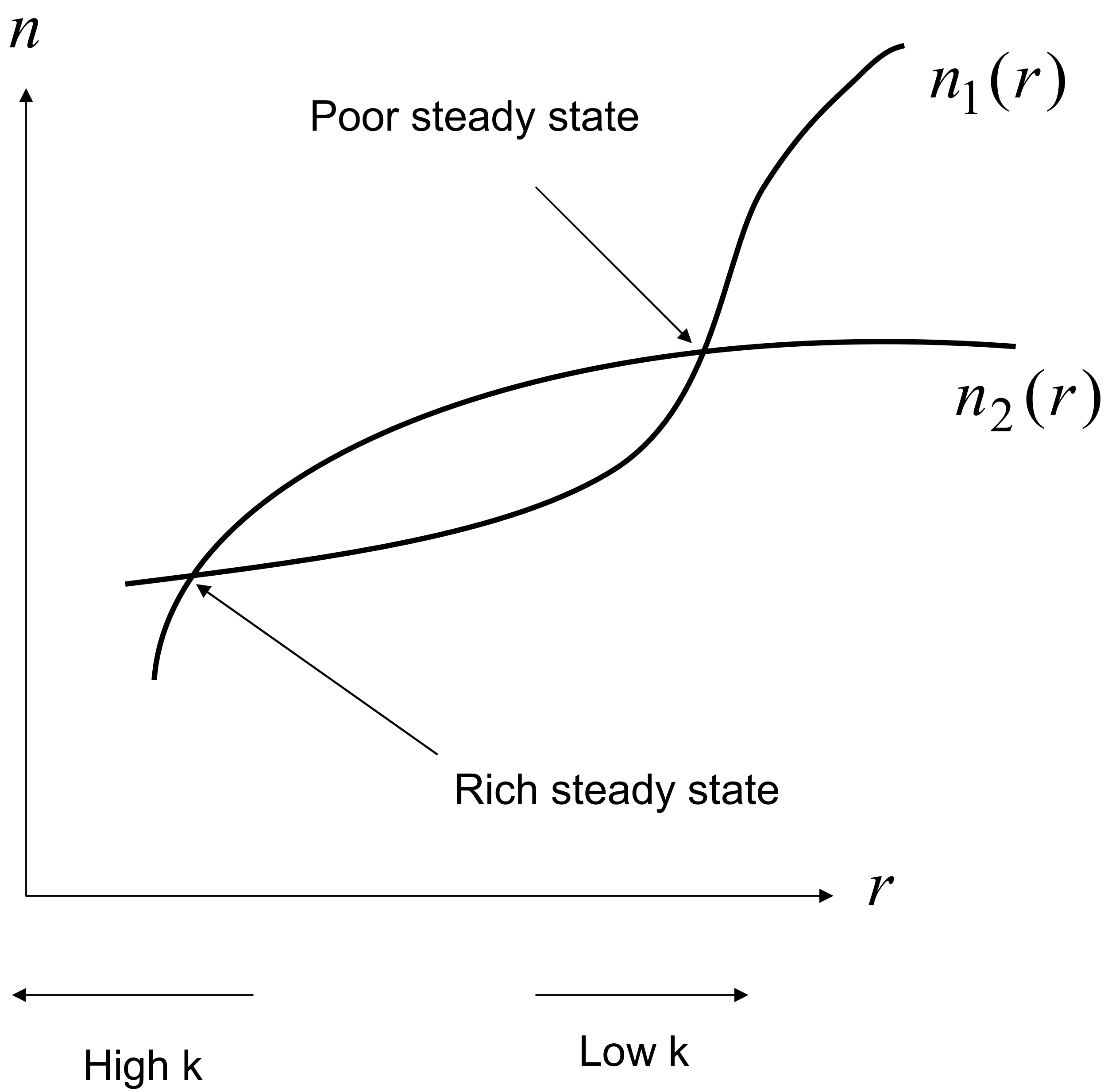
Upward sloping $n_2(r)$ makes multiple steady state equilibria possible

Poor steady state: low k, high r, high n

Rich steady state: high k, low r, low n

Consistent with cross-country observations

Intuition: low k, low wages, cheap children, high quantity/low quality of children, this sustains low k through low bequests


n
Poor steady state
$n_1(r)$
$n_2(r)$
Rich steady state
r
High k
Low k

Endogenous Fertility with Human Capital

Often cited paper: Becker, Murphy, and Tamura (1990)

Different sections of the paper – different stories

Here: focus on Section III

Barro-Becker type of preferences:

$$V_t = \frac{1}{\sigma}(c_t)^\sigma + \alpha n_t^{1-\varepsilon} V_{t+1} \tag{22}$$

$H_t =$ human capital of generation t

Production function for human capital invested in children:

$$H_{t+1} = A h_t [H^0 + H_t] \tag{23}$$

$h_t =$ time spent educating each child

$H^0 =$ minimum level of labor productivity

Budget constraint:

$$c_t = [1 - \{v + h_t\}n_t]\,[H^0 + H_t] - fn_t \qquad (24)$$

Using (23):

$$c_t = [1 - vn_t]\,[H^0 + H_t] - \frac{H_{t+1}n_t}{A} - fn_t \qquad (25)$$

Costs of children:
a fixed amount of v units of time (plus h_t in education)
a fixed goods cost, f

Bellman equation:

$$V(H_t) = \qquad (26)$$

$$\max_{n_t, H_{t+1}} \left\{ \begin{array}{c} \frac{1}{\sigma}\left(\overbrace{\begin{array}{c} [1 - vn_t]\,[H^0 + H_t] \\ -\frac{H_{t+1}n_t}{A} - fn_t \end{array}}^{c_t} \right)^{\sigma} \\ \\ +\alpha n_t^{1-\varepsilon} V(H_{t+1}) \end{array} \right\}$$

First-order condition for H_{t+1} depends on whether constraint that $h_t \geq 0$ is binding or not

If $h_t > 0$ in optimum:

$$\alpha n_t^{1-\varepsilon} V'(H_{t+1}) = \frac{n_t c_t^{\sigma-1}}{A} \tag{27}$$

If $h_t = 0$ in optimum, the marginal utility of consuming today must be greater than the marginal benefit of educating children:

$$\alpha n_t^{1-\varepsilon} V'(H_{t+1}) < \frac{n_t c_t^{\sigma-1}}{A} \tag{28}$$

To find Euler Equation, use Envelope:

$$V'(H_t) = c_t^{\sigma-1}[1 - v n_t] + 0 \tag{29}$$

Forward this one period, and use either (27) or (28) – this gives the Euler Equation:

$$\frac{n_t^{\varepsilon}}{\alpha} \left(\frac{c_{t+1}}{c_t} \right)^{1-\sigma} \geq A[1 - v n_{t+1}] \tag{30}$$

where inequality is strict if $h_t = 0$, and takes equality if $h_t > 0$

First-order condition for n_t:

$$(1-\varepsilon)\alpha n_t^{-\varepsilon} V(H_{t+1}) \\ = c_t^{\sigma-1}\left[(v+h_t)[H^0+H_t]+f\right] \tag{31}$$

Hard to find explicit difference equation (H_{t+1} in terms of H_t)

Alternative approach: check if a steady state exists where $H_t = 0$, and examine it properties; then check if balanced growth path exists and examine its properties

Malthusian steady state

Search for steady state where education time is zero, i.e., for all t: $h_t = h_{t+1} = H_t = H_{t+1} = 0$

Called Malthusian steady state (after Thomas Malthus) – poverty trap with high fertility

Variables referring to Malthusian steady state labelled with sub-index u

Since $h_t = 0$ the Euler Eq. in (30) takes strict inequality; evaluated in steady state it becomes

$$n_u^{\varepsilon} > \alpha A[1 - vn_u] \tag{32}$$

Next see if the endogenous n_u is such that (32) holds; see first-order condition for fertility in (31), evaluated at $H_{t+1} = H_t = h_t = 0$

$$\begin{aligned} &(1-\varepsilon)\alpha n_u^{-\varepsilon} V(0) \\ &= c_u^{\sigma-1}\left[vH^0 + f\right] \end{aligned} \tag{33}$$

Next find $V(0)$ – use the value function in (22):

$$V(0) = \frac{1}{\sigma}(c_u)^{\sigma} + \alpha n_u^{1-\varepsilon} V(0) \tag{34}$$

or

$$V(0) = \frac{\frac{1}{\sigma}(c_u)^{\sigma}}{1 - \alpha n_u^{1-\varepsilon}} \tag{35}$$

Together (33) and (35) give

$$\frac{c_u}{vH^0 + f} = \frac{\sigma\left[1 - \alpha n_u^{1-\varepsilon}\right]}{(1-\varepsilon)\alpha n_u^{-\varepsilon}} \tag{36}$$

Then use the consumption budget constraint in (24) imposing $H_{t+1} = H_t = h_t = 0$

$$c_u = [1 - v n_u] H^0 - f n_u \tag{37}$$

Insert (37) into (36); we now have an expression which implicitly defines n_u:

$$\frac{[1 - v n_u] H^0 - f n_u}{vH^0 + f} = \frac{\sigma\left[1 - \alpha n_u^{1-\varepsilon}\right]}{(1-\varepsilon)\alpha n_u^{-\varepsilon}} \tag{38}$$

or

$$\frac{H^0}{vH^0 + f} - n_u = \frac{\sigma}{(1-\varepsilon)\alpha}\left[n_u^\varepsilon - \alpha n_u\right] \qquad (39)$$

LHS decreasing in n_u; RHS hump shaped in n_u – existence of n_u not guaranteed (in fact, not even uniqueness)

However, for some parameter values it holds that:
(a) there is some (feasible) n_u at which (39) holds
(b) the Malthusian steady state condition in (32) holds: $n_u^\varepsilon > \alpha A[1 - vn_u]$

Local stability also holds: if the $h_t \geq 0$ constraint binds at $H_t = 0$, it must do so for some sufficiently small $H_t > 0$; thus $H_{t+1} = 0$ next period

Balanced growth path

On the balanced growth path (BGP) some variables grow at a sustained rate: e.g. c_t, H_t; denote the growth rate g^*

Others are constant in levels, denoted by superscript *: e.g. n^*, h^*

Use budget constraint in (25) to see that on the BGP

$$\frac{c_t}{H_t} \to [1 - vn^*] - \frac{(1+g^*)\,n^*}{A} \tag{40}$$

implying c_t and H_t must grow at same rate, g^*

Use (29) to see that on BGP

$$\frac{V'(H_{t+1})}{V'(H_t)} = \left(\underbrace{c_{t+1}/c_t}_{1+g^*}\right)^{\sigma-1} \tag{41}$$

For H_t and c_t to grow at same rate, the BGP value function must take the functional form:

$$V(H_t) = \text{const} \times H_t^{\sigma} \tag{42}$$

Or:

$$\frac{V'(H_t)H_t}{V(H_t)} = \sigma \tag{43}$$

Use (27) and (31), set $h_t = h^*$, let $H_t \to \infty$, and $H_{t+1} = Ah^*H_t$

$$\frac{1}{1-\varepsilon}\underbrace{\left(\frac{V'(H_{t+1})H_{t+1}}{V(H_{t+1})}\right)}_{\sigma} = \frac{Ah^*}{v+h^*} \tag{44}$$

Solving for h^* gives:

$$h^* = \frac{\sigma v}{1-\varepsilon-\sigma} \tag{45}$$

and

$$1 + g^* = \lim_{H_t \to \infty} \frac{H_{t+1}}{H_t} = Ah^* = \frac{A\sigma v}{1 - \varepsilon - \sigma} \qquad (46)$$

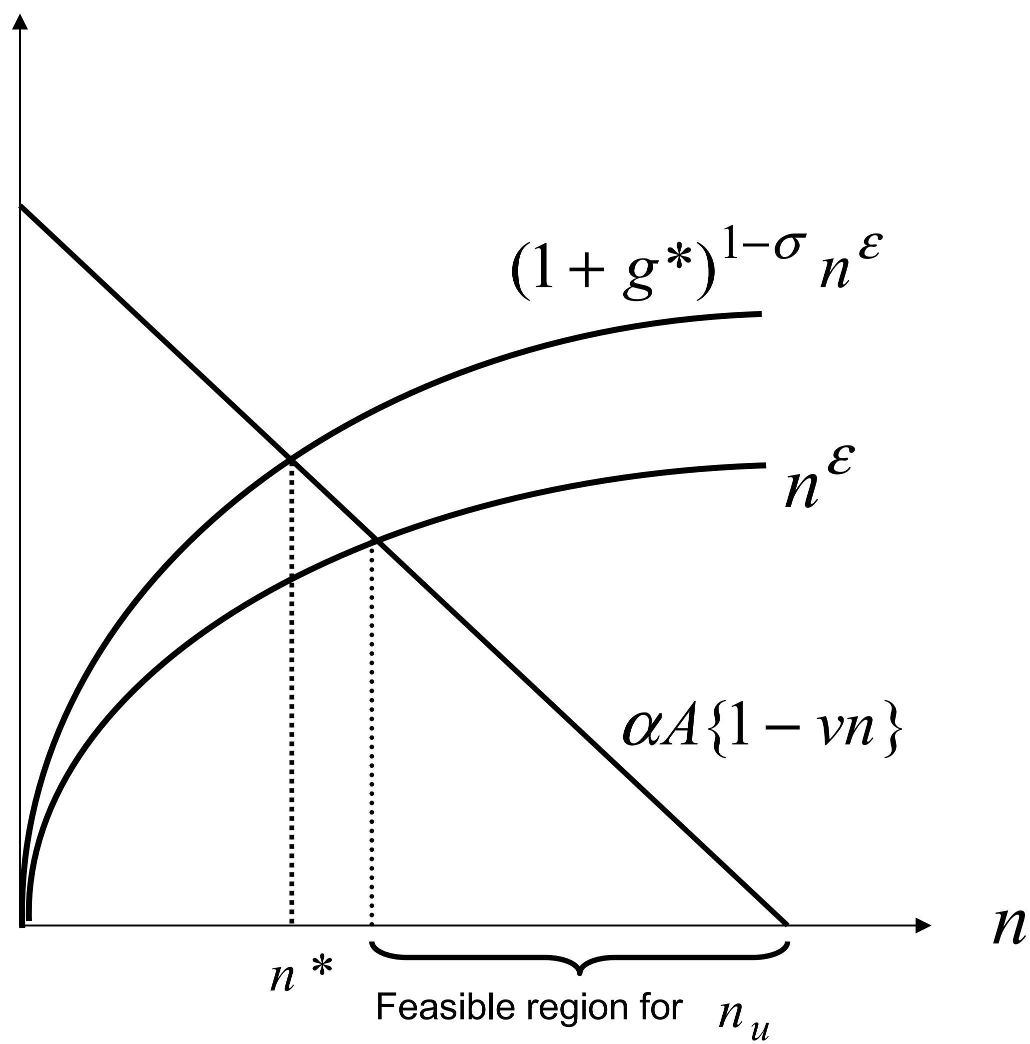
BGP exists if RHS$>$1

From Euler equation in (30):

$$(n^*)^{\varepsilon} \left(\underbrace{1 + g^*}_{= \frac{A\sigma v}{1-\varepsilon-\sigma}} \right)^{1-\sigma} = \alpha A[1 - vn^*] \qquad (47)$$

defines n^* uniquely

Using (47) and (32), we see that BGP fertility is less than Malthusian fertility


$(1+g^*)^{1-\sigma} n^{\varepsilon}$
n^{ε}
$\alpha A\{1-vn\}$
n
n^*
Feasible region for n_u

Endogenous fertility and gender heterogeneity

Here: Galor and Weil (1996)

Two types of labor: mental labor, L_t^m (brains), and physical labor, L_t^p (brawns)

Women have only brains; men have brains and brawns

Production:

$$Y_t = K_t^{\alpha} \left(L_t^m \right)^{1-\alpha} + bL_t^p \tag{48}$$

Crucial feature of production function: rising capital stock, K_t, means higher returns to brains relative to brawns, and thus smaller gender gap in potential earnings

Intensive form: lower-case variables denote per-physical-labor units

$$\begin{aligned} y_t &= k_t^{\alpha} m_t^{1-\alpha} + b \\ m_t &= \frac{L_t^m}{L_t^p} \\ k_t &= \frac{K_t}{L_t^p} \end{aligned} \quad (49)$$

where m_t is mental-over-physical labor; k_t is the capital-physical-labor ratio

Wages:

$$\begin{aligned} w_t^m &= (1-\alpha) k_t^{\alpha} m_t^{-\alpha} \\ w_t^p &= b \end{aligned} \quad (50)$$

Men's and women's wages:

$$\begin{aligned} \text{men: } & w_t^m + b \\ \text{women: } & w_t^m \end{aligned} \quad (51)$$

Time cost per child $= z$; $n_t = \#$ of children

Household income if $zn_t \leq 1$:

$$w_t^m + b + w_t^m[1 - zn_t] = (2w_t^m + b) - w_t^m zn_t \tag{52}$$

women's labor time $= 1 - zn_t$; man's labor time $= 1$

Household income if $zn_t \geq 1$:

$$[w_t^m + b]\,(2 - zn_t) = 2\,(w_t^m + b) - (b + w_t^m)zn_t \tag{53}$$

women's labor time $= 0$; man's labor time $= 1 - (zn_t - 1)$

Consumption only in old age; income $=$ saving $= s_t$; consumption $=$ saving plus interest:

$$c_{t+1} = s_t(1 + r_{t+1}) \tag{54}$$

Trade-off between s_t and zn_t different on the margin depending on zn_t:

$$s_t = \begin{cases} (2w_t^m + b) - w_t^m zn_t & \text{if } zn_t \leq 1 \\ 2(w_t^m + b) - (b + w_t^m) zn_t & \text{if } zn_t \geq 1 \end{cases} \tag{55}$$

Or:

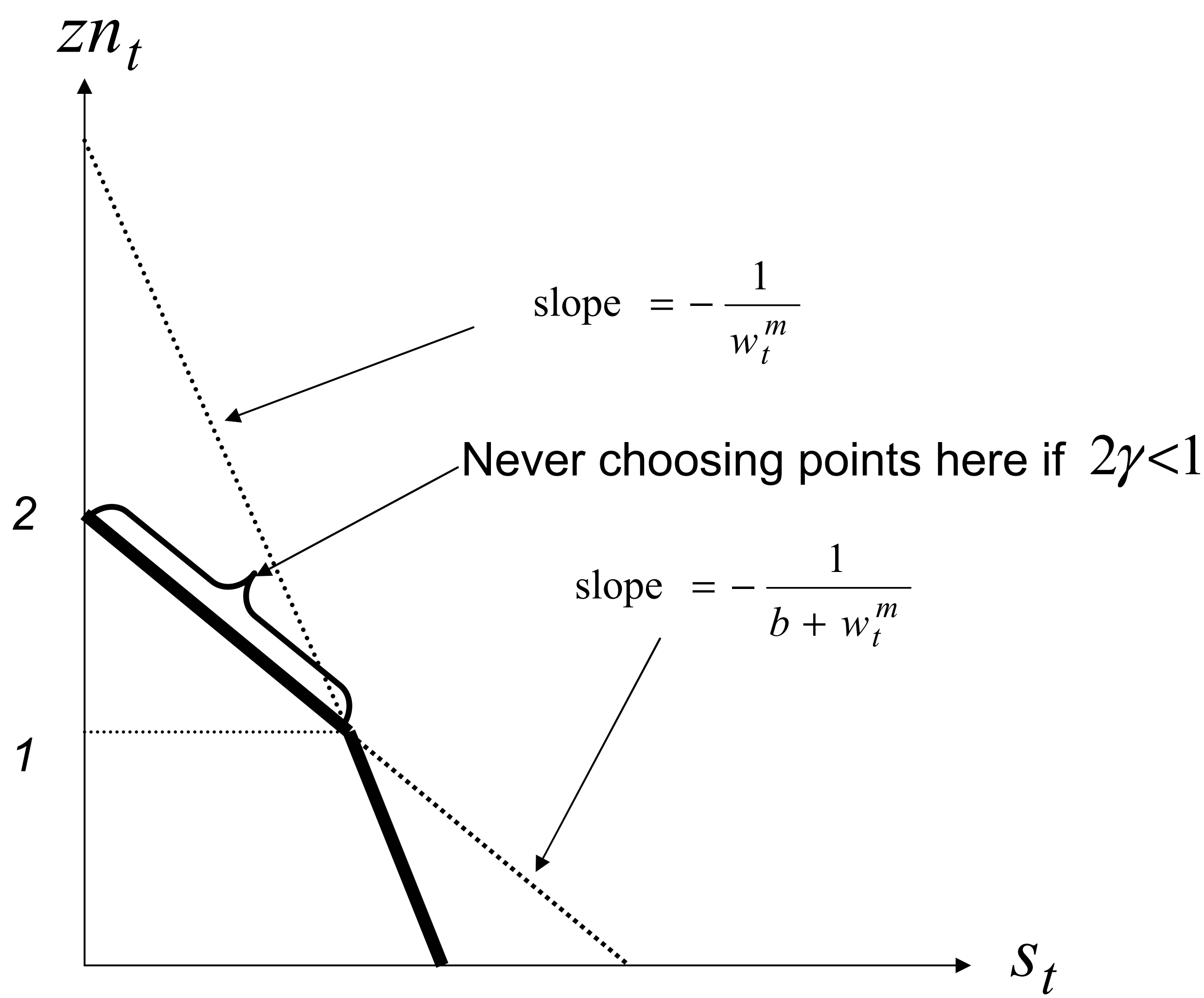
$$zn_t = \begin{cases} \frac{2w_t^m + b}{w_t^m} - \frac{s_t}{w_t^m} & \text{if } zn_t \leq 1 \\ 2 - \frac{s_t}{w_t^m + b} & \text{if } zn_t \geq 1 \end{cases} \tag{56}$$

Utility:

$$u_t = \gamma \ln n_t + (1 - \gamma) \ln c_{t+1} \tag{57}$$

If $zn_t \geq 1$, FOC would give $zn_t = 2\gamma$

To see this, set up the maximization problem conjec-


zn_t
slope $= -\frac{1}{w_t^m}$
Never choosing points here if $2\gamma<1$
2
slope $= -\frac{1}{b + w_t^m}$
1
s_t

turing that $zn_t \geq 1$:

$$\max_{zn_t} \overbrace{\gamma \ln(zn_t) - \gamma \ln(z)}^{\gamma \ln(n_t)}$$

$$+(1-\gamma) \ln \left[(1 + r_{t+1}) \{ \underbrace{2\,(w_t^m + b) - (b + w_t^m) zn_t \}}_{s_t} \right]$$

FOC gives:

$$zn_t = \gamma \frac{2\,(w_t^m + b)}{(b + w_t^m)} = 2\gamma$$

Assume $\gamma < 1/2$: implies $zn_t \leq 1$ in optimum; $zn_t > 1$ can never hold

Father never stays home taking care of children; the household is either in a corner solution where the mother stays home and the father works ($zn_t = 1$), or mother works some of the time and father full time ($zn_t < 1$)

Optimal fertility:

$$zn_t = \begin{cases} \gamma \left[2 + \frac{b}{w_t^m}\right] & \text{if } \gamma \left[2 + \frac{b}{w_t^m}\right] \leq 1 \\ 1 & \text{if } \gamma \left[2 + \frac{b}{w_t^m}\right] \geq 1 \end{cases} \tag{58}$$

Intuition: the unconstrained (non-corner) choice of n_t such that $n = \gamma\times$ income/(price per child). Income $= 2w_t^m + b$; price per child = mother's wage (w_t^m) $\times$ z

Mental labor supply: $m_t = 2 - zn_t$

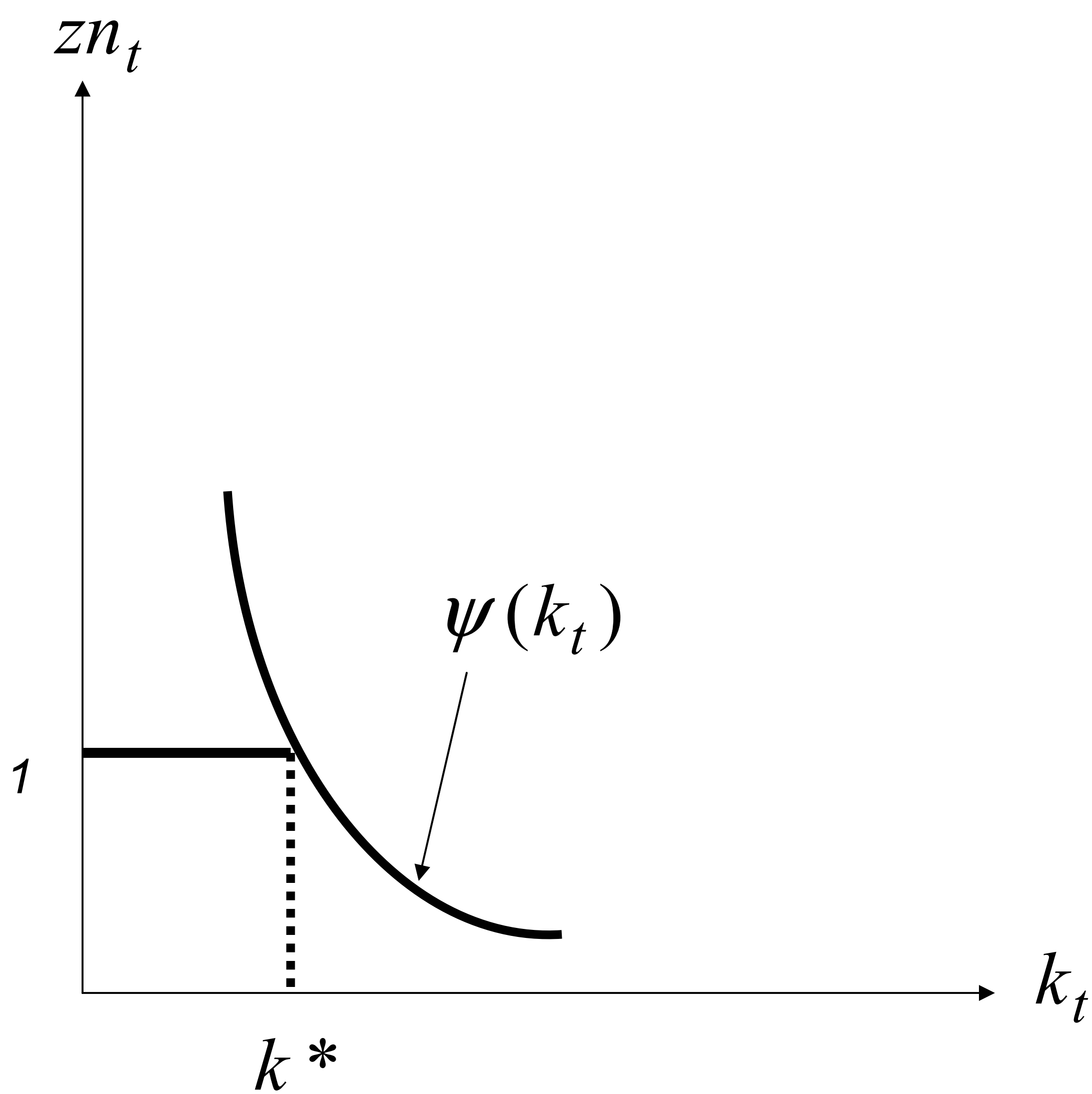
We can then write time spent on children as $zn_t = \psi(k_t)$, where $\psi(k_t)$ is defined from

$$\psi(k_t) = \gamma \left[2 + \frac{b}{\underbrace{(1-\alpha)k_t^{\alpha}[2-\psi(k_t)]^{-\alpha}}_{w_t^m}}\right] \tag{59}$$

Or:

$$\psi(k_t) = \gamma \left[2 + \frac{b[2 - \psi(k_t)]^\alpha}{(1-\alpha)k_t^\alpha} \right] \tag{60}$$

Note that $\psi'(k_t) < 0$ (see problem set)



zn_t
$\psi(k_t)$
1
k_t
k^*

Exercise/idea for a paper

Set $\alpha = 1/2$

Define $\xi(k_t) = [2 - \psi(k_t)]^{1/2} = [2 - \psi(k_t)]^{\alpha}$

Then we can use (60) to write:

$$\xi^2 = 2 - \psi(k_t) = 2 - \gamma \left[2 + \frac{b2\xi}{k_t^{\alpha}}\right]$$

This can be solved for explicitly for ξ, and thus for $\psi(k_t)$

Possible to simulate time paths

Define k^* as the level of k_t above which the fertility choice gets out of the corner (mother starts working); i.e., $\psi(k^*) = 1$

Use (60): $1 = \gamma \left[2 + \frac{b \times 1}{(1-\alpha)(k^*)^\alpha}\right]$; this gives

$$k^* = \left\{\frac{b\gamma}{(1-\alpha)(1-2\gamma)}\right\}^{\frac{1}{\alpha}} \tag{61}$$

Fertility given by

$$n_t = \left(\frac{1}{z}\right) \min\{1, \psi(k_t)\}$$

Difference equation for k_t

Consider $k_t \leq k^*$ and $k_t \geq k^*$ separately

(1) $k_t \leq k^*$

$zn_t = 1$

$m_t = 2 - zn_t = 1$

$w_t^m = (1-\alpha)k_t^\alpha m_t^{-\alpha} = (1-\alpha)k_t^\alpha$
$s_t = w_t^m + b = (1-\alpha)k_t^\alpha + b$

$$\begin{aligned} k_{t+1} &= \frac{s_t}{(n_t/2)} = \frac{2zs_t}{zn_t} = 2zs_t \\ &= 2z[(1-\alpha)k_t^\alpha + b] \equiv \phi^0(k_t) \end{aligned} \tag{62}$$

(2) $k_t \geq k^*$
$zn_t = \gamma \frac{2w_t^m + b}{w_t^m}$
$s_t = 2w_t^m + b - zn_t w_t^m = (1-\gamma)[2w_t^m + b]$
$k_{t+1} = \frac{s_t}{(n_t/2)} = 2z\left(\frac{s_t}{zn_t}\right) = 2z\left(\frac{1-\gamma}{\gamma}\right) w_t^m$
Use definition of $\psi(k_t)$ in (59): $w_t^m = \frac{b\gamma}{\psi(k_t) - 2\gamma}$
Thus:

$$k_{t+1} = \frac{2zb(1-\gamma)}{\psi(k_t) - 2\gamma} \equiv \phi^1(k_t) \tag{63}$$

Jointly:

$$k_{t+1} = \begin{cases} \phi^0(k_t) & \text{if } k_t \leq k^* \\ \phi^1(k_t) & \text{if } k_t \geq k^* \end{cases} \tag{64}$$

To see how $\phi^0(k_t)$ and $\phi^1(k_t)$ are positioned, note the following:

From (59): $\psi(0) = \infty$, implying that $\phi^1(0) = 0$

From (62): $\phi^0(0) = 2zb > 0$; that is: $\phi^0(0)$ starts off above $\phi^1(0)$

How about when $k_t > 0$? First note from the definition of $\psi(k_t)$ in (59) that

$$\frac{(1-\alpha)k_t^\alpha}{b} = \frac{\gamma\left\{2 - \psi(k_t)\right\}^\alpha}{\psi(k_t) - 2\gamma} \tag{65}$$

Next use def's of $\phi^0(k_t)$ and $\phi^1(k_t)$

$$\phi^0(k_t) > (=, <)\phi^1(k_t)$$

$$\Longleftrightarrow$$

$$2z[(1-\alpha)k_t^\alpha + b] > (=, <)\tfrac{2zb(1-\gamma)}{\psi(k_t)-2\gamma}$$

$$\Longleftrightarrow$$

$$(1-\alpha)k_t^\alpha > (=, <)b\left\{\tfrac{(1-\gamma)}{\psi(k_t)-2\gamma} - 1\right\}$$

$$\Longleftrightarrow$$

$$\tfrac{(1-\alpha)k_t^\alpha}{b} > (=, <)\tfrac{(1-\gamma)-\psi(k_t)+2\gamma}{\psi(k_t)-2\gamma}$$

$$\Longleftrightarrow$$

$$\tfrac{\gamma\{2-\psi(k_t)\}^\alpha}{\psi(k_t)-2\gamma} > (=, <)\tfrac{(1-\gamma)-\psi(k_t)+2\gamma}{\psi(k_t)-2\gamma}$$

$$\Longleftrightarrow$$

$$\gamma\left\{2-\psi(k_t)\right\}^\alpha > (=, <)(1-\gamma) - \psi(k_t) + 2\gamma$$

$$= 2 - \psi(k_t) - (1-\gamma)$$

Three cases:

(1) If $k_t > k^*$ it holds that $\psi(k_t) < 1$; thus $2 - \psi(k_t) > 1$, and $\{2 - \psi(k_t)\}^\alpha < 2 - \psi(k_t)$ (since $\alpha < 1$); so $\gamma \{2 - \psi(k_t)\}^\alpha < \gamma [2 - \psi(k_t)] < 2 - \psi(k_t) - (1 - \gamma)$ (where last inequality comes from $(1 - \gamma) < [2 - \psi(k_t)] (1 - \gamma)$, since $\psi(k_t) < 1$); thus: $\underline{k_t > k^* \text{ implies } \phi^0(k_t) > \phi^1(k_t).}$

(2) If $k_t < k^*$ it holds that $\psi(k_t) > 1$; thus $2 - \psi(k_t) < 1$, and $\{2 - \psi(k_t)\}^\alpha > 2 - \psi(k_t)$ (since $\alpha < 1$); so $\gamma \{2 - \psi(k_t)\}^\alpha > \gamma [2 - \psi(k_t)] > 2 - \psi(k_t) - (1 - \gamma)$ (where last inequality comes from $(1 - \gamma) > [2 - \psi(k_t)] (1 - \gamma)$, since $\psi(k_t) > 1$); thus: $\underline{k_t < k^* \text{ implies } \phi^0(k_t) < \phi^1(k_t).}$

(3) If $k_t = k^*$ it holds that $\psi(k_t) = 1$, which analogously to (1) and (2) means that $\phi^0(k_t) = \phi^1(k_t)$. Thus: $\underline{k_t = k^* \text{ implies } \phi^0(k_t) = \phi^1(k_t).}$

We can thus write the difference equation in (64) as:

$$k_{t+1} = \max\{\phi^0(k_t), \phi^1(k_t)\}$$

Illustrate dynamics in phase diagram; here focus on case where there is a unique steady state $\overline{k} > k^*$; multiple steady states also possible

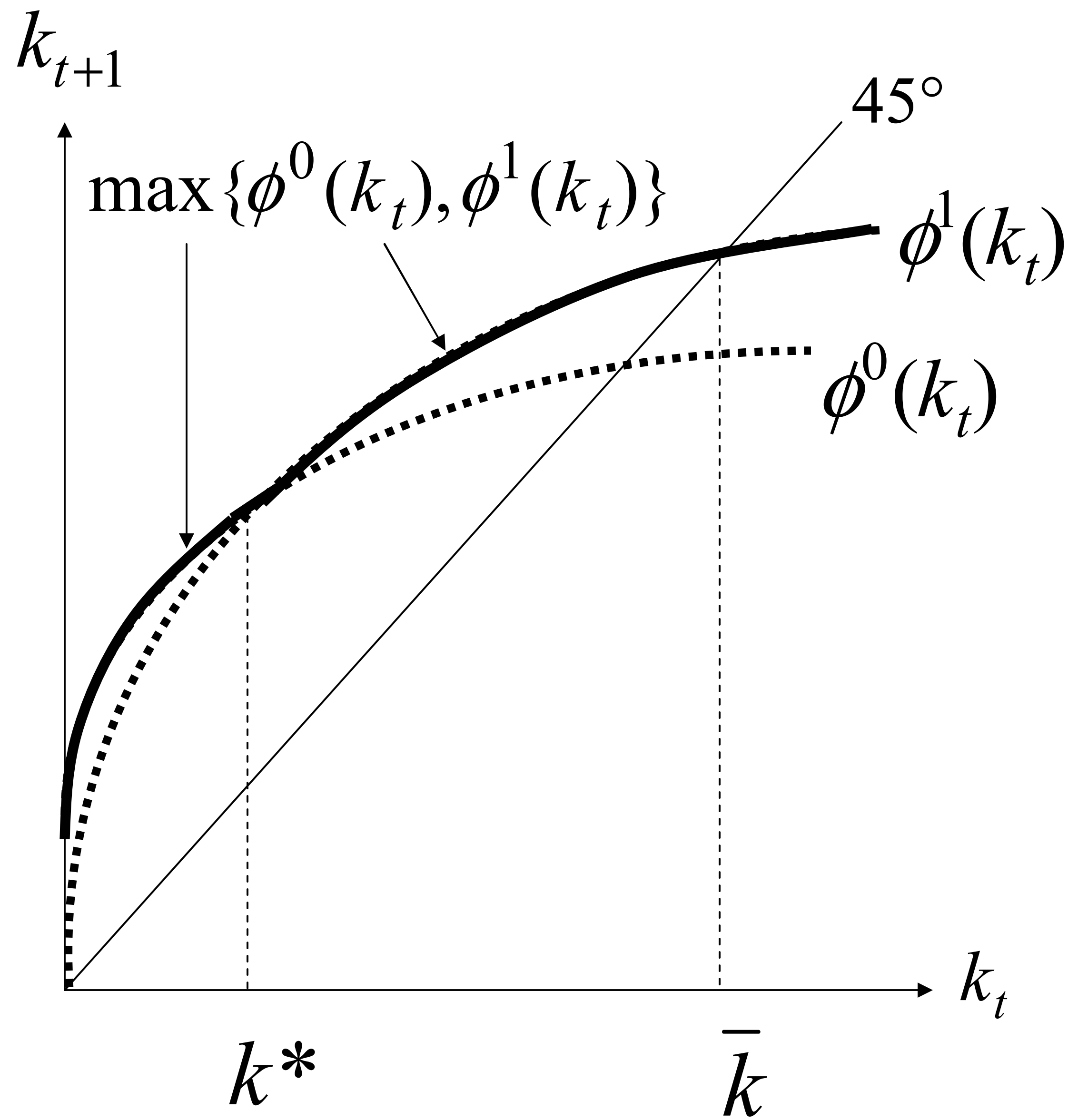
Time path for capital stock: spurt at k^*

Time path for fertility: constant at 1, then starts to fall as $k_t > k^*$

Time path for female labor supply: constant at 0, then starts to increase as $k_t > k^*$

Consistent with the experience of many countries: fertility falls as women go out on the labor market

Other observations: small changes in parameter values can change the dynamics from a multiple steady states configuration, to a "spurt" configuration


k_{t+1}
45°
$\max\{\phi^0(k_t), \phi^1(k_t)\}$
$\phi^1(k_t)$
$\phi^0(k_t)$
k_t
k^*
$\bar{k}$