Lecture Notes in Growth Theory - Part I Preliminary and under updating

Nils-Petter Lagerlöf
Department of Economics
York University

January 16, 2006

Big questions in economics:

Why are some countries poor, and others rich?

All rich countries *used* to be poor hundreds of years ago: what made them grow rich?

Some countries used to be relatively rich, but have been overtaken by previously poor countries: why?

What types of policies/institutions make countries grow rich?

Why are these policies/institutions not already in place?

Some Facts:

USA is one of the richest countries in the world today

Richest here means: highest GDP per capita, but most other measures give similar picture

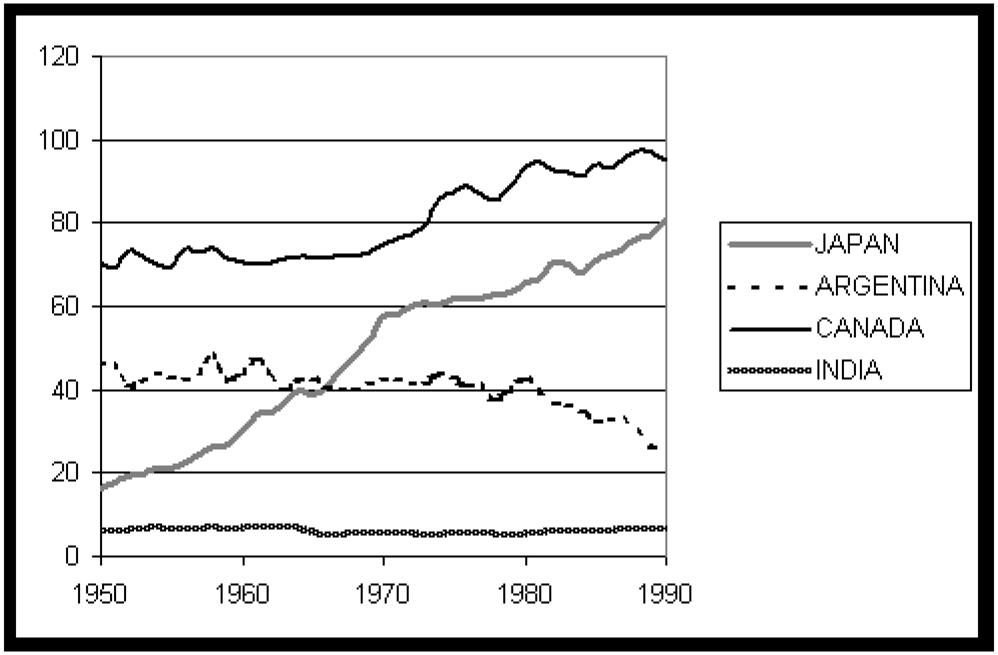
4 examples for comparison: Japan, India, Argentina, Canada

Compare their GDP/capita relative to the US, 1950-90

See figure: comparison of four countries 1950-1990

Per-capita incomes relative to USA

Canada: catching up
Japan: remarkable spurt
Argentina: stagnating
India: little change since 1950



120
100
80
60
40
20
0
1950
1960
1970
1980
1990
JAPAN
ARGENTINA
CANADA
INDIA

Income gaps over time and across regions

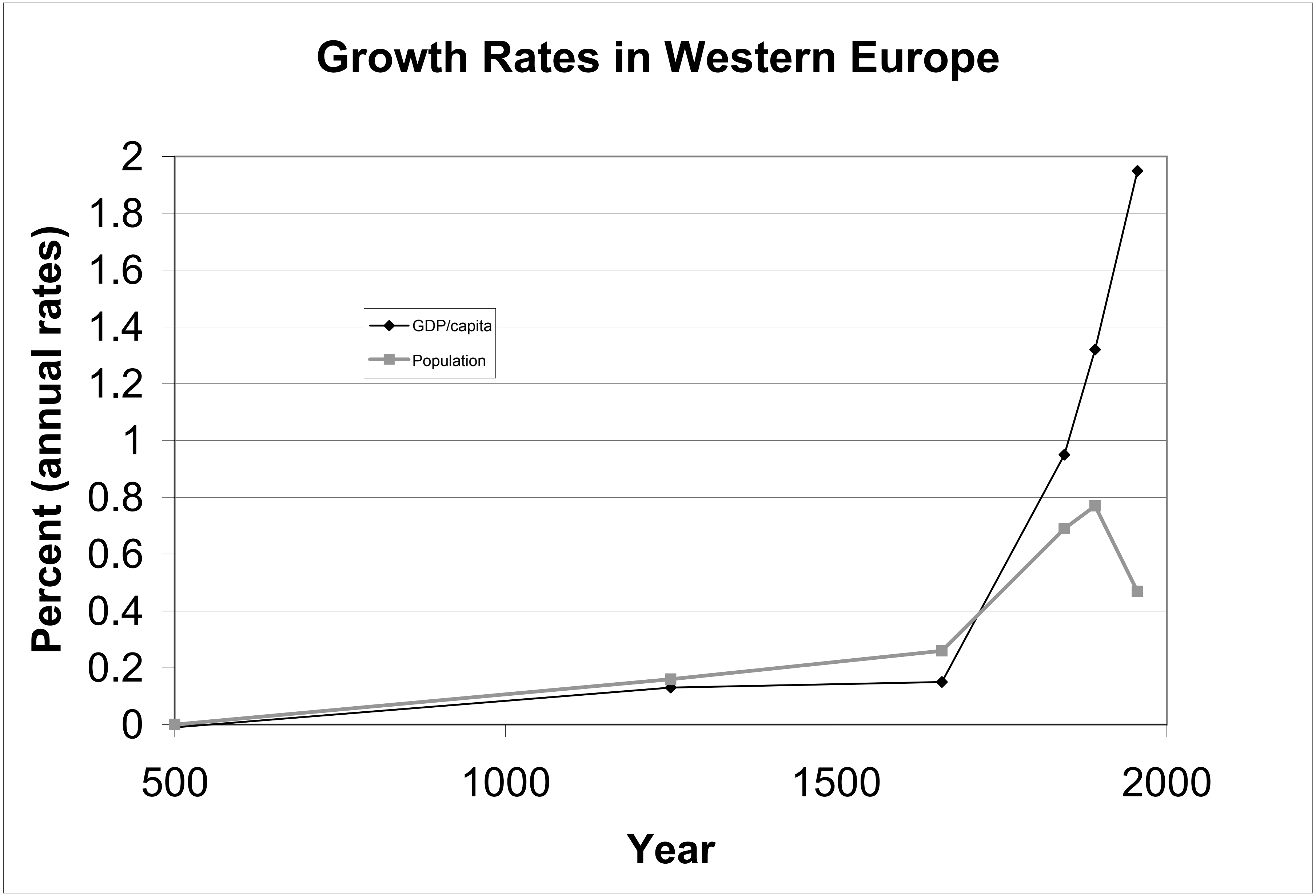
- Across regions:
  - Poor countries in (Sub-Saharan) Africa, Latin America, (South/Central) Asia
  - Rich countries in North America, Europe, Australia, East Asia

- Over time:
  - We were *all* poor 1,000 years ago, or more
  - A few despotic rulers excepted
  - Sustained growth in per-capita income started only (some) 200 years ago: called *Industrial Revolution*
  - At the same time: increased *population growth*

– But later a decline in population growth

– Called the *Demographic Transition*

– Switch from growth in # people to growth in living standards


Growth Rates in Western Europe
Percent (annual rates)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
GDP/capita
Population
500
1000
1500
2000
Year

- Also: changes over time in cross-regional income gaps
  - First "pristine" *civilizations* on earth in Middle East (Mesopotamia), Egypt, China, India, Mesoamerica (the Aztecs/Maya), South America (the Incas) — not richest today
  - "Reversal of fortune" issue: Climate/geography?
  - Why was Western Europe first to experience an Industrial Revolution? "Guns, Germs & Steel"

Many variables correlated with income:

- Poor countries usually have:
  - Lower levels of education
  - More child labor
  - Higher (child/infant) mortality rates
  - Higher fertility rates
  - Less gender equality
  - More conflicts (e.g. civil wars, revolutions)
  - More corruption, less property rights, worse roads...
- Why these patterns? Cause, effect?

Growth Models

- Many questions: how can we start looking for answers?

- Set up a model!!!

- Starting point: high income levels due to (long) *periods of growth* in income levels. No jumps over night (cf Figure)

- What makes income levels *grow over time?*

- Time component calls for *dynamic* models

- Means: *many* periods (typically infinitely many)

How does this course differ from other growth courses?

- More on some specific topics: fertility, mortality, gender equality, institutions, "very long run" growth. Less on other topics: R&D, technological diffusion, barriers-to-riches literature, growth accounting

- More theory, less empirical work/econometrics
  - But we'll talk about data, and do some quantitative exercises

- Many growth courses use *continuous time* models; here only (or mostly) *discrete time*

How course is organized: 3 components

- These lecture notes
  - think of it as textbook, but with more typos
  - available of my home page
- Problem sets
  - not to be handed in

- Reading list
  - a set of journal articles, some book chapters – to be updated; see my home page
  - get published articles online via library home page:
    http://www.library.yorku.ca/ search on journal name
  - unpublished papers (mimeos): can often be found online, google on author/title; may put link on web site
  - some book chapters also in library for copying: search at library web site under Course reserve material, instructor name Lagerloef, course name "Theory of Growth-Soc'lst. Econ (ECON5380)" (I know, don't ask)
  - some material is hard to read; use as reference; good training in academic communication

Basics: terminology etc.

Growth models are dynamic models: involve time, t

Two types of dynamic models:

Discrete-time models: Means the variable t (time) is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Discrete-time variables usually written x_t

Described by difference equations: $x_{t+1} = \phi(x_t)$

Continuous-time models: Means t is a (non-negative) real number: $t \in [0, +\infty) = \Re_+$

Continuous-time variables usually written $x(t)$

Described by differential equations : $\overset{\bullet}{x}(t) = \frac{\partial x(t)}{\partial t} = \zeta(x(t))$

Here: emphasis on discrete-time (versions of) growth models

Difference equations

Some terminology

Autonomous: $x_{t+1} = \phi(x_t)$
non-autonomous: $x_{t+1} = \phi(x_t, t)$

1-dimensional: $x_{t+1} = \phi(x_t)$
2-dimensional: $x_{t+1} = \phi(x_t, y_t)$, $y_{t+1} = \psi(x_t, y_t)$

Linear: $x_{t+1} = a + bx_t$
non-linear: $x_{t+1} = \phi(x_t)$, $\phi''(x_t) \neq 0$

1st order: $x_{t+1} = \phi(x_t)$
2nd order: $x_{t+2} = \phi(x_{t+1}, x_t)$

Exercise: write 2nd order, 1-dimensional difference equation, $x_{t+2} = \phi(x_{t+1}, x_t)$, as 1st order, 2-dimensional

Solution: let $y_t \equiv x_{t+1}$; this gives: $y_{t+1} = \phi(y_t, x_t)$, and $x_{t+1} = y_t$.

Solutions to difference equations:

$x_{t+1} = \phi(x_t)$ has solution $\{x_t\}_{t=0}^{\infty}$

That is: $\{x_t\}_{t=0}^{\infty}$ is solution to $x_{t+1} = \phi(x_t)$ if $x_{t+1} = \phi(x_t)$ for all $t \in \{0, 1, 2...\}$

Intuitive idea: start with x_0, which gives $x_1 = \phi(x_0)$, which gives $x_2 = \phi(x_1)$, and so on.

Steady state equilibrium (or steady state):

$\overline{x}$ is a steady state to $x_{t+1} = \phi(x_t)$ if, and only if, $\overline{x} = \phi(\overline{x})$

I.e., $\overline{x}$ is a *fixed point* to $\phi(x)$

Stability

$\overline{x}$ is locally stable if, and only if:

$\phi'(\overline{x}) \in (-1, 1)$

(locally stable here roughly same as "asymptotically" locally stable)

$\overline{x}$ is globally stable if x_t converges to $\overline{x}$ regardless of starting value, x_0

Usually equivalent to $\overline{x}$ being (a) locally stable, and (b) unique

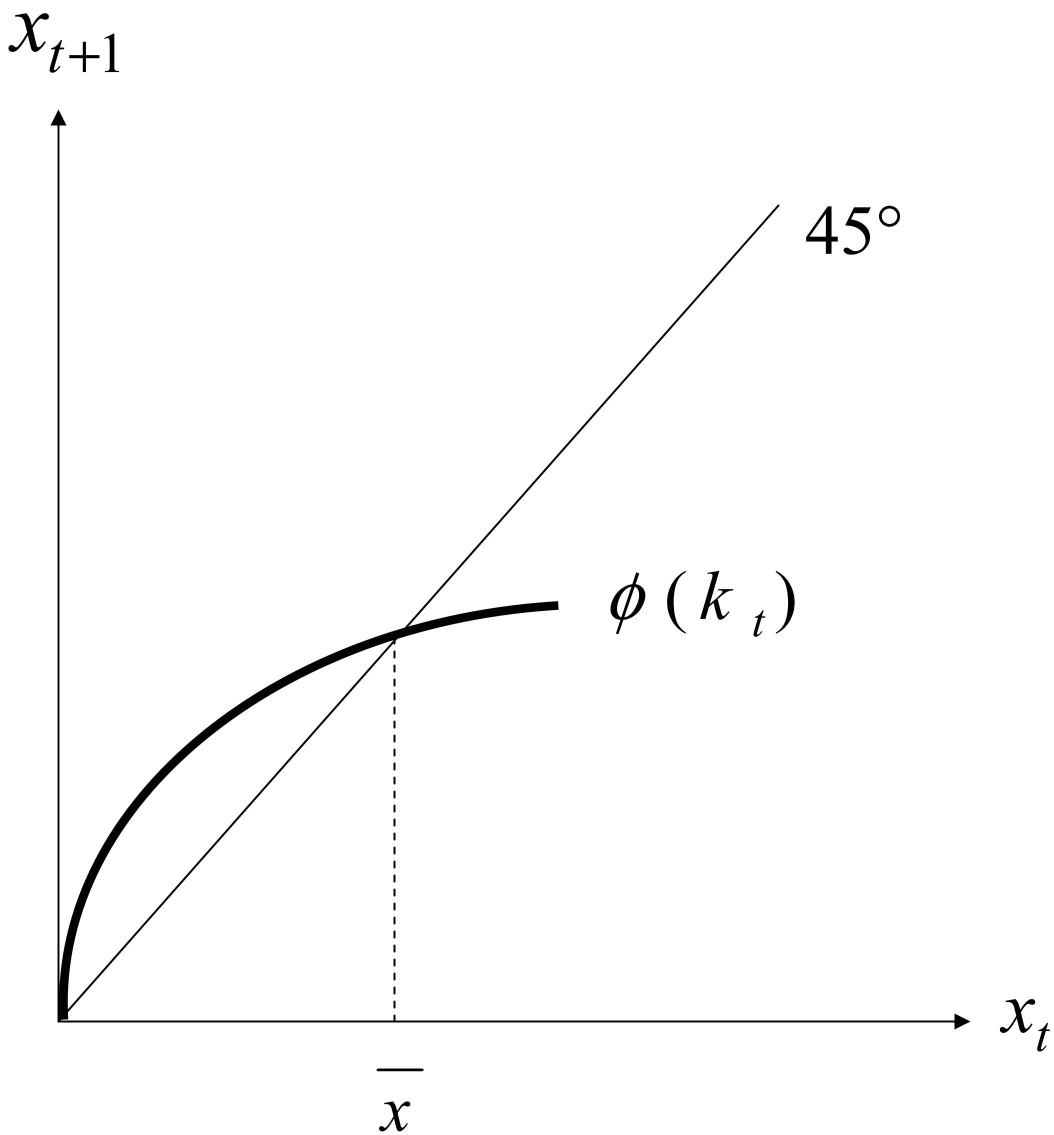
If $\phi'(\overline{x}) < 0$ the steady state is called *oscillatory*

Locally stable steady state: $\phi'(\overline{x}) \in (0,1)$

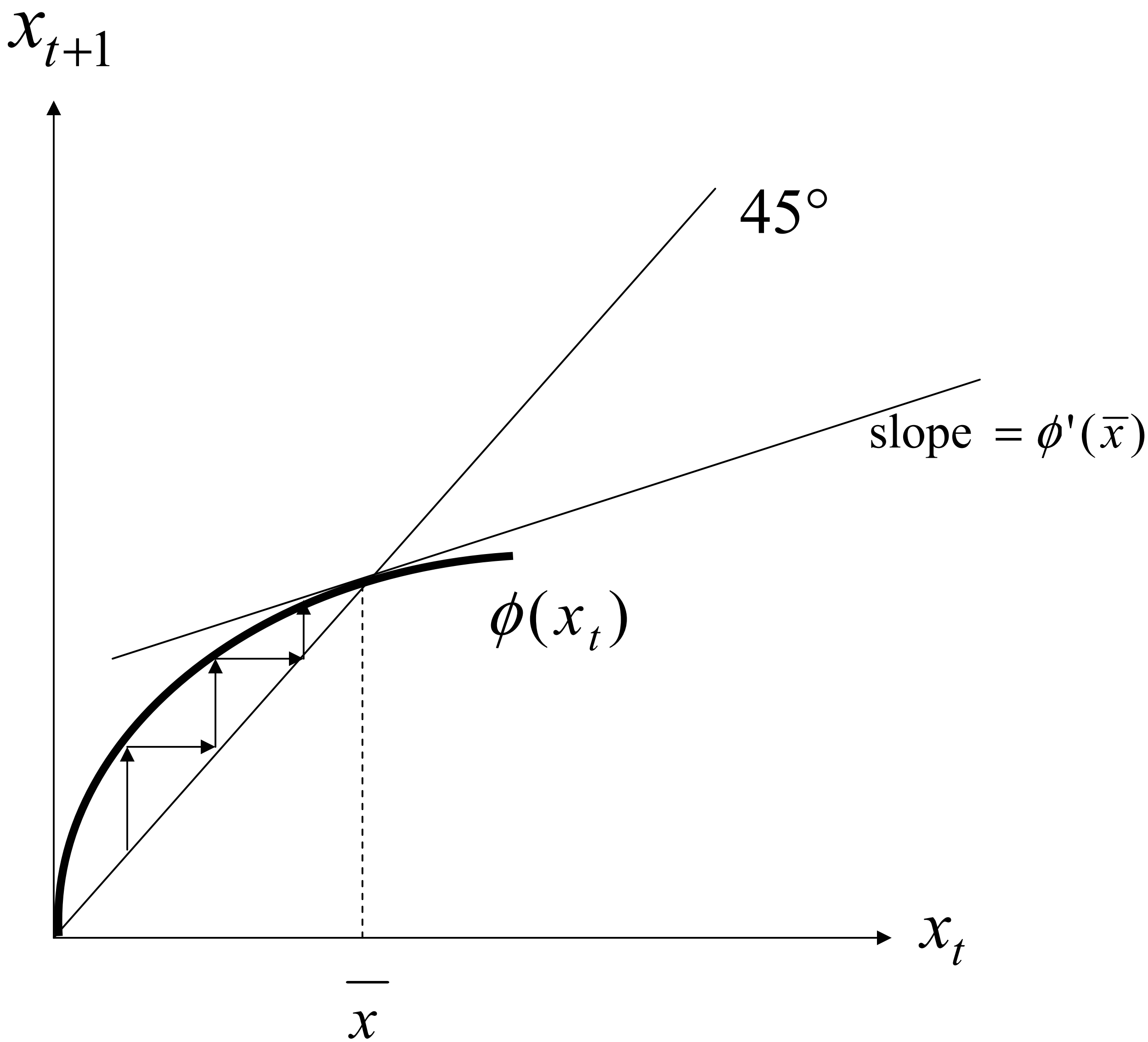
Locally unstable steady state: $\phi'(\overline{x}) > 1$

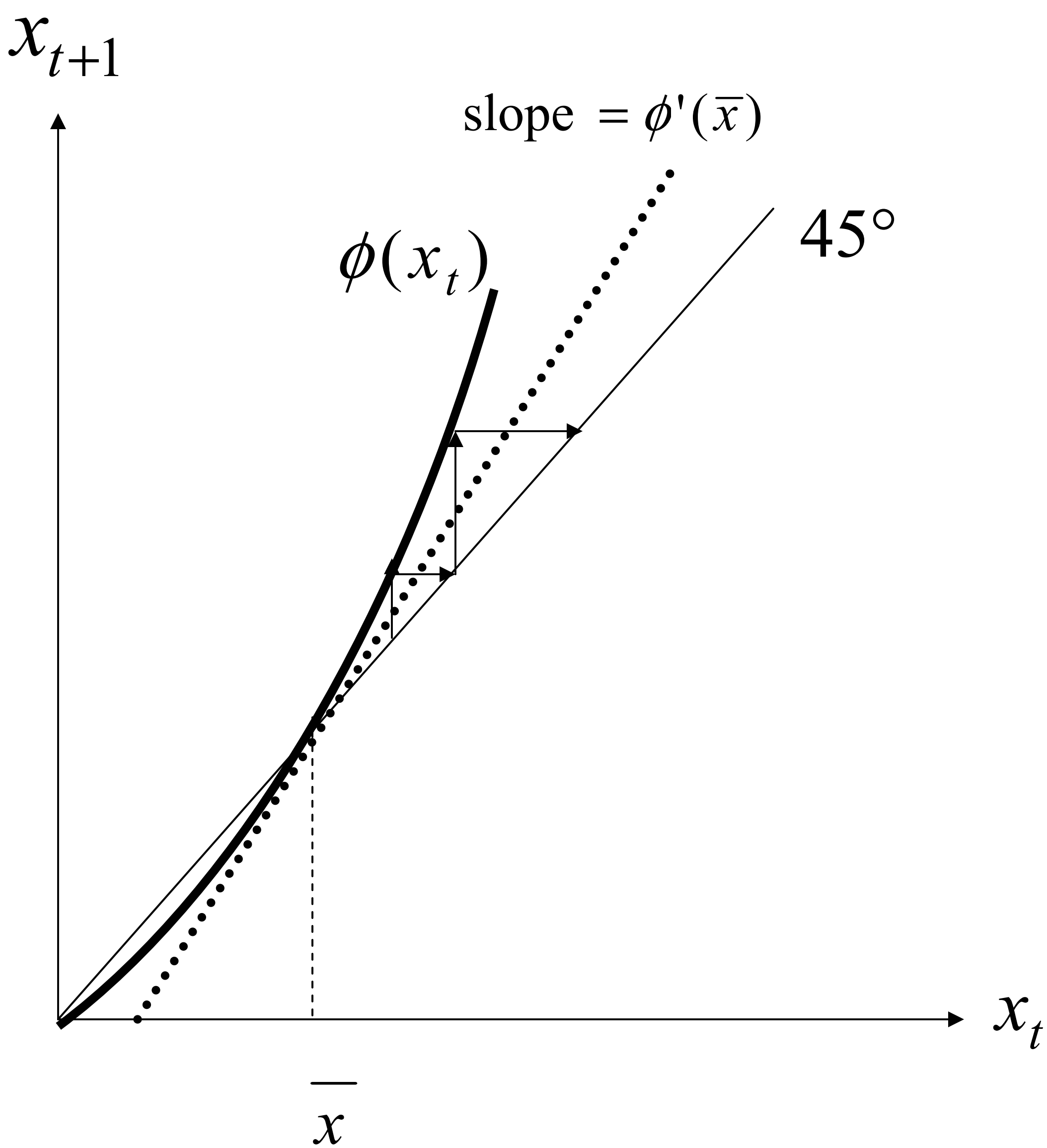
Oscillatory locally stable steady state: $\phi'(\overline{x}) \in (-1,0)$

Oscillatory locally non-stable steady state: $\phi'(\overline{x}) < -1$

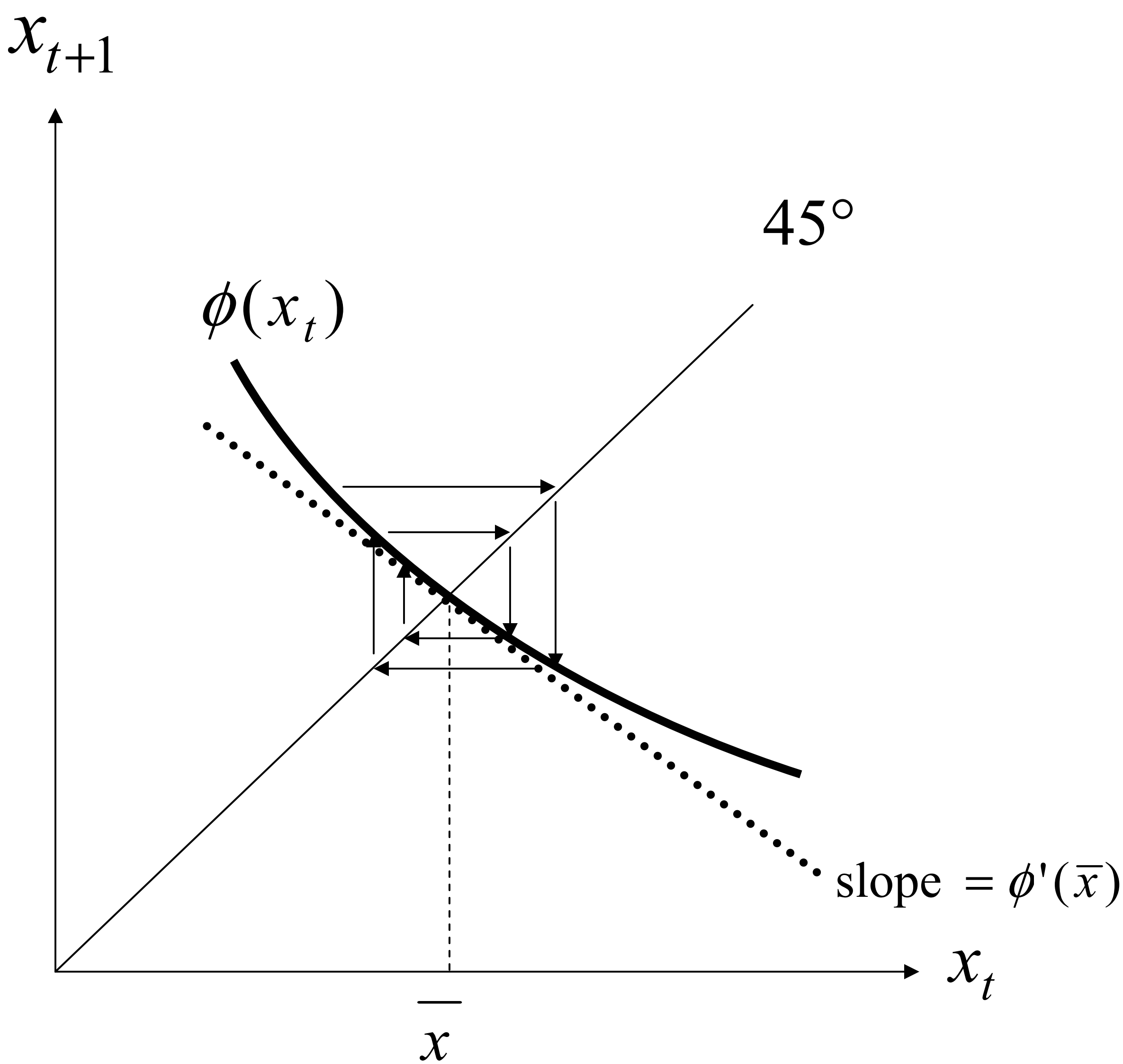


x_{t+1}
45°
$\phi(k_t)$
x_t
$\bar{x}$


x_{t+1}
45°
slope $= \phi'(\bar{x})$
$\phi(x_t)$
x_t
$\bar{x}$



x_{t+1}
slope $= \phi'(\bar{x})$
$45°$
$\phi(x_t)$
x_t
$\bar{x}$



x_{t+1}
45°
$\phi(x_t)$
slope $= \phi'(\bar{x})$
x_t
$\bar{x}$

Production functions

Standard setting: Y=output, K (capital) and L (labor)= inputs

$$Y = F(K, L)$$

Neo-classical production functions satisfy these conditions:

(1) Positive marginal products:

$$F_K(\cdot) > 0, F_L(\cdot) > 0$$

(2) Diminishing marginal products:

$$F_{KK}(\cdot) < 0, F_{LL}(\cdot) < 0$$

(3) The Inada condition:

$$\lim_{Z \to 0} F_Z(\cdot) = \infty, \lim_{Z \to \infty} F_Z(\cdot) = 0, \; Z = K, L$$

(4) Constant Returns to Scale (CRS):

$$\lambda F(K,L) = F(\lambda K, \lambda L), \text{ for all } \lambda > 0$$

Intensive form production function

Set $\lambda = 1/L$

Let lower-case variables denote per-worker levels

$$\begin{aligned} y &= \frac{Y}{L} = \frac{F(K,L)}{L} = \\ &= F\left(\frac{K}{L}, 1\right) = F(k,1) \equiv f(k) \end{aligned}$$

Assumptions (1)-(3) above imply:

$$\begin{aligned} f'(k) &> 0, f''(k) < 0 \\ \lim_{k \to 0} f'(k) &= \infty, \lim_{k \to \infty} f'(k) = 0 \end{aligned}$$

Also, using l'Hôpital's Rule:

$$\lim_{k \to 0} \frac{f(k)}{k} = \lim_{k \to 0} \frac{f'(k)}{1} = \infty$$

Factor prices

Atomistic firms take factor prices as given when maximizing profits:

$$\max_{K,L} F(K,L) - wL - (r+\delta)K$$

w=wage rate, r=real interest rate, δ=depreciation rate

Rewrite $F(K,L) = Lf(k) = Lf(\frac{K}{L})$

Exercise: show that

$$\begin{aligned} w &= F_L(K,L) = f(k) - f'(k)k \\ r &= F_K(K,L) - \delta = f'(k) - \delta \end{aligned}$$

I.e., K and L paid their marginal products

If $\delta = 1$ (full depreciation): gross interest rate given by $R \equiv 1 + r = f'(k)$

Parametric examples of production functions

Cobb-Douglas:

$$\begin{aligned} F(K, L) &= K^{\alpha} L^{1-\alpha} \\ f(k) &= k^{\alpha} \end{aligned}$$

CES (various formulations):

$$\begin{aligned} F(K, L) &= [\alpha K^{\sigma} + (1-\alpha) L^{\sigma}]^{\frac{1}{\sigma}} \\ f(k) &= [\alpha k^{\sigma} + (1-\alpha)]^{\frac{1}{\sigma}} \end{aligned}$$

where $\sigma \in (-\infty, 1]$

Alternative formulations: $\rho \equiv -\sigma \in [-1, \infty)$

The Solow Growth Model

Discrete time setting

Notation: K_t = capital in period t; δ = depreciation rate; s = rate of saving/investment out of income, Y_t

Evolution of capital stock:

$$K_{t+1} = sY_t + (1 - \delta)K_t \tag{1}$$

L_t = population/labor force in period t

n = growth rate of population

$$L_{t+1} = (1 + n)L_t \tag{2}$$

Assume $n \geq 0$, $\delta \geq 0$

Income:

$$Y_t = F(K_t, L_t) \tag{3}$$

Find difference equation for $k_t = K_t/L_t$, on the form: $k_{t+1} = \phi(k_t)$

Use (1) to (3) to get

$$\begin{aligned} \tfrac{K_{t+1}}{L_t} = \tfrac{K_{t+1}}{L_{t+1}}\tfrac{L_{t+1}}{L_t} = k_{t+1}(1+n) \\ = \tfrac{sY_t+(1-\delta)K_t}{L_t} \\ = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t \end{aligned} \tag{4}$$

Or:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{5}$$

Properties of $\phi(k_t)$

$$\phi'(k_t) = \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} f'(k_t) > 0$$

$$\phi''(k_t) = \tfrac{s}{1+n} f''(k_t) < 0$$

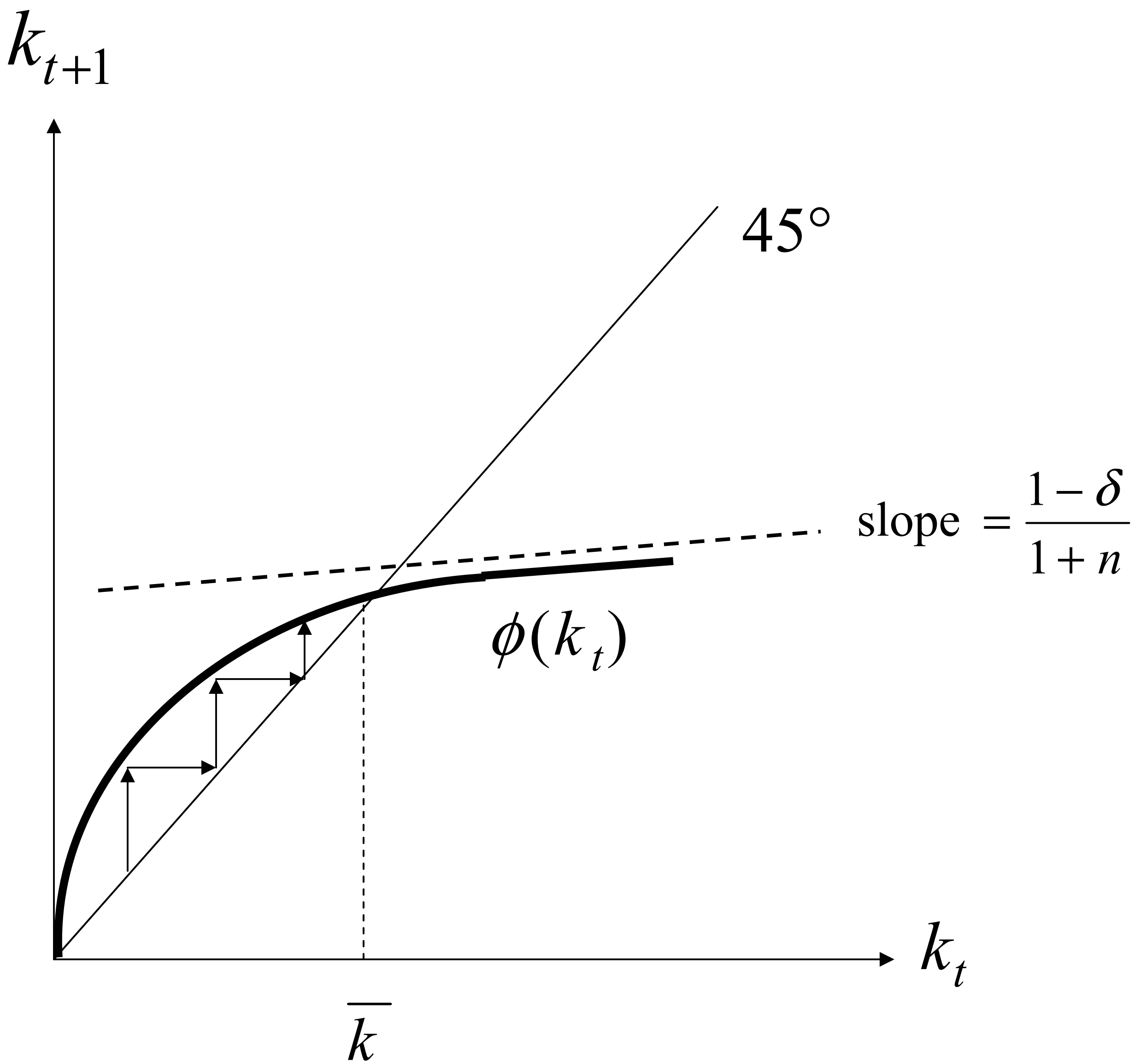
$$\lim_{k_t \to 0} \phi'(k_t) = \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to 0} f'(k_t) = \infty \qquad (6)$$

$$\lim_{k_t \to \infty} \phi'(k_t) = \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to \infty} f'(k_t)$$
$$= \tfrac{1-\delta}{1+n} < 1 \quad (\text{unless } n = \delta = 0)$$

Together these guarantee: existence, uniqueness, and stability of steady state – see 45°-diagram

Check: If one of the 4 properties above change, how come existence, uniqueness, and stability are no longer guaranteed?

Problem sets: relax assumptions about (a) neoclassical production function; and (b) saving rate (s) being same across factor payments



k_{t+1}
45°
slope $= \frac{1-\delta}{1+n}$
$\phi(k_t)$
k_t
$\overline{k}$

The Diamond Overlapping Generations Model

Agents live in two periods: working age, retirement

$c_{1,t} =$ consumption of working agent in period t

$c_{2,t} =$ consumption of retired agent in period t

$s_t =$ saving of working agent in period t

$R_{t+1} = 1 + r_{t+1} =$ gross interest rate on savings held from period t to $t + 1$

$w_t =$ period-t wage rate

Consider agent young/working in period t

Working-age budget constraint: $c_{1,t} = w_t - s_t$

Old-age budget constraint: $c_{2,t+1} = R_{t+1}s_t$

Utility: $U_t = U(c_{1,t}, c_{2,t+1})$

Optimal savings decision given by $s(w_t, R_{t+1})$, defined from

$$s(w_t, R_{t+1}) = \arg\max_{s_t \in [0, w_t]} U(w_t - s_t, R_{t+1}s_t) \tag{7}$$

Example: logarithmic utility

$$\max_{s_t \in [0, w_t]} (1 - \beta) \ln(w_t - s_t) + \beta \ln(R_{t+1}s_t) \tag{8}$$

First-order condition:

$$-(1 - \beta)(w_t - s_t)^{-1} + \beta s_t^{-1} = 0 \tag{9}$$

Solving for s_t gives:

$$s(w_t, R_{t+1}) = \beta w_t \tag{10}$$

Capital accumulation

Total savings in period t = total capital stock in period $t+1$

$$s_t L_t = K_{t+1} \tag{11}$$

L_t = number young people in period t

$L_{t+1} = (1+n)L_t$

$$\begin{aligned} \frac{K_{t+1}}{L_t} &= k_{t+1}(1+n) \\ &= s_t = s(w_t, R_{t+1}) \end{aligned} \tag{12}$$

Factor prices (recall) given by marginal products

$$\begin{aligned} w_t &= f(k_t) - f'(k_t)k_t \equiv w(k_t) \\ R_{t+1} &= f'(k_{t+1}) + 1 - \delta \equiv R(k_{t+1}) \end{aligned} \tag{13}$$

Thus, $k_{t+1} = \phi(k_t)$ where $\phi(k_t)$ is defined from

$$\phi(k_t) = \frac{s\{w(k_t), R(\phi(k_t))\}}{1+n} \tag{14}$$

Note: at given k_t, k_{t+1} is not necessarily unique. To see this, hold k_t constant, and replace $\phi(k_t)$ in (14) with k_{t+1}, i.e., $k_{t+1} = s\{w(k_t), R(k_{t+1})\}/(1+n)$

Always true that $R'(k_{t+1}) < 0$. Assume $\partial s(\cdot)/\partial R_{t+1} < 0$ (over some interval)

Then the right-hand side is increasing in k_{t+1}

Since the left-hand side is too, there could be multiple k_{t+1} for given k_t

Intuition: if savings are high, interest rates are low (=marginal product of capital), sustaining high savings [given $\partial s(\cdot)/\partial R_{t+1} < 0$]

Called an *indeterminacy;* implies backward bending $\phi(k_t)$

Illustration: see 45°-diagram

Logarithmic example again: $s(w_t, R_{t+1}) = \beta w_t$

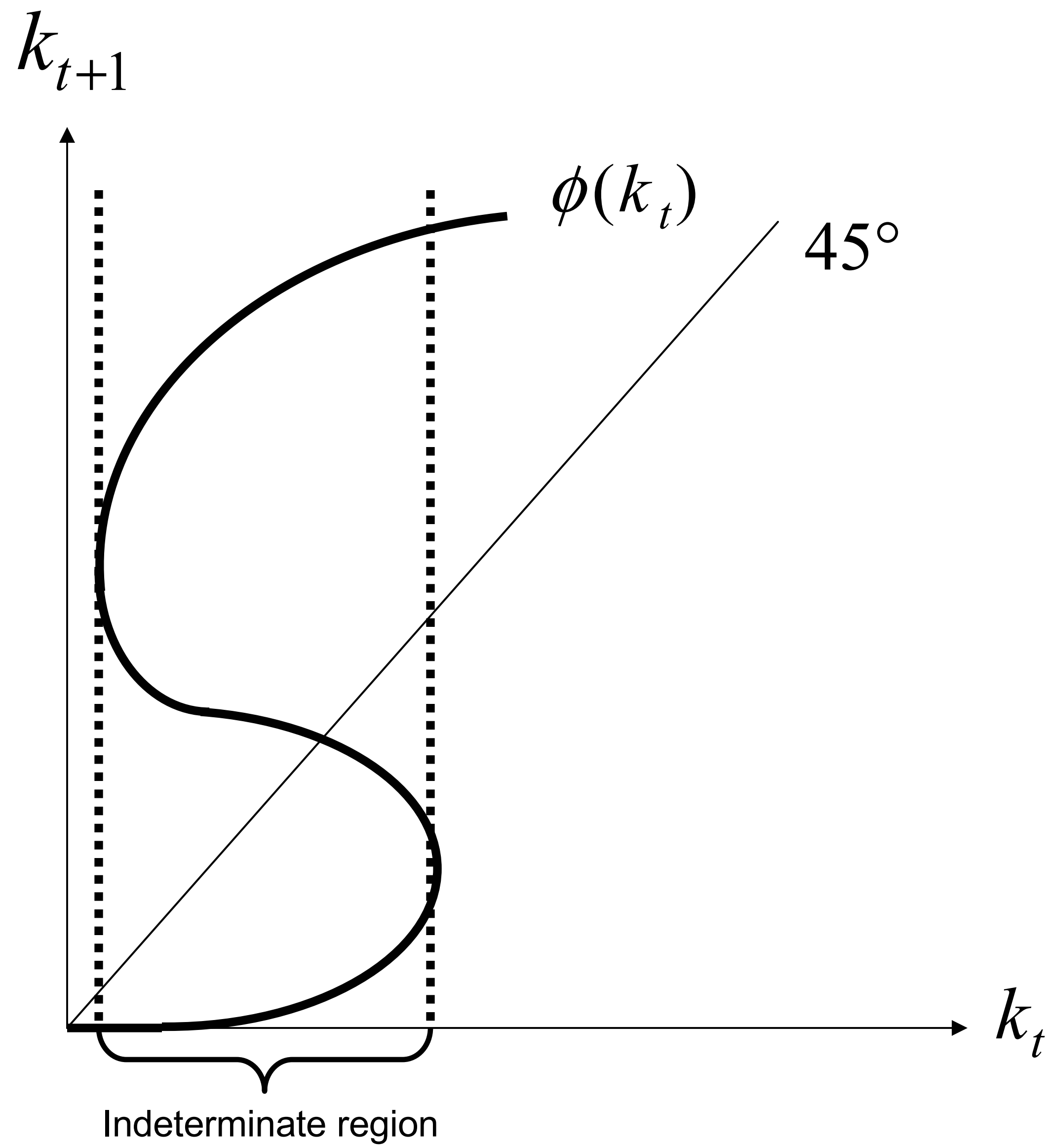
$$k_{t+1} = \frac{\beta\left[f(k_t) - f'(k_t)k_t\right]}{1+n} \equiv \phi(k_t) \tag{15}$$

$$\begin{aligned} \phi'(k_t) &= \tfrac{-\beta}{1+n} f''(k_t)k_t > 0 \\ \phi''(k_t) &= \tfrac{-\beta}{1+n} f'''(k_t) \gtrless 0 \end{aligned} \tag{16}$$

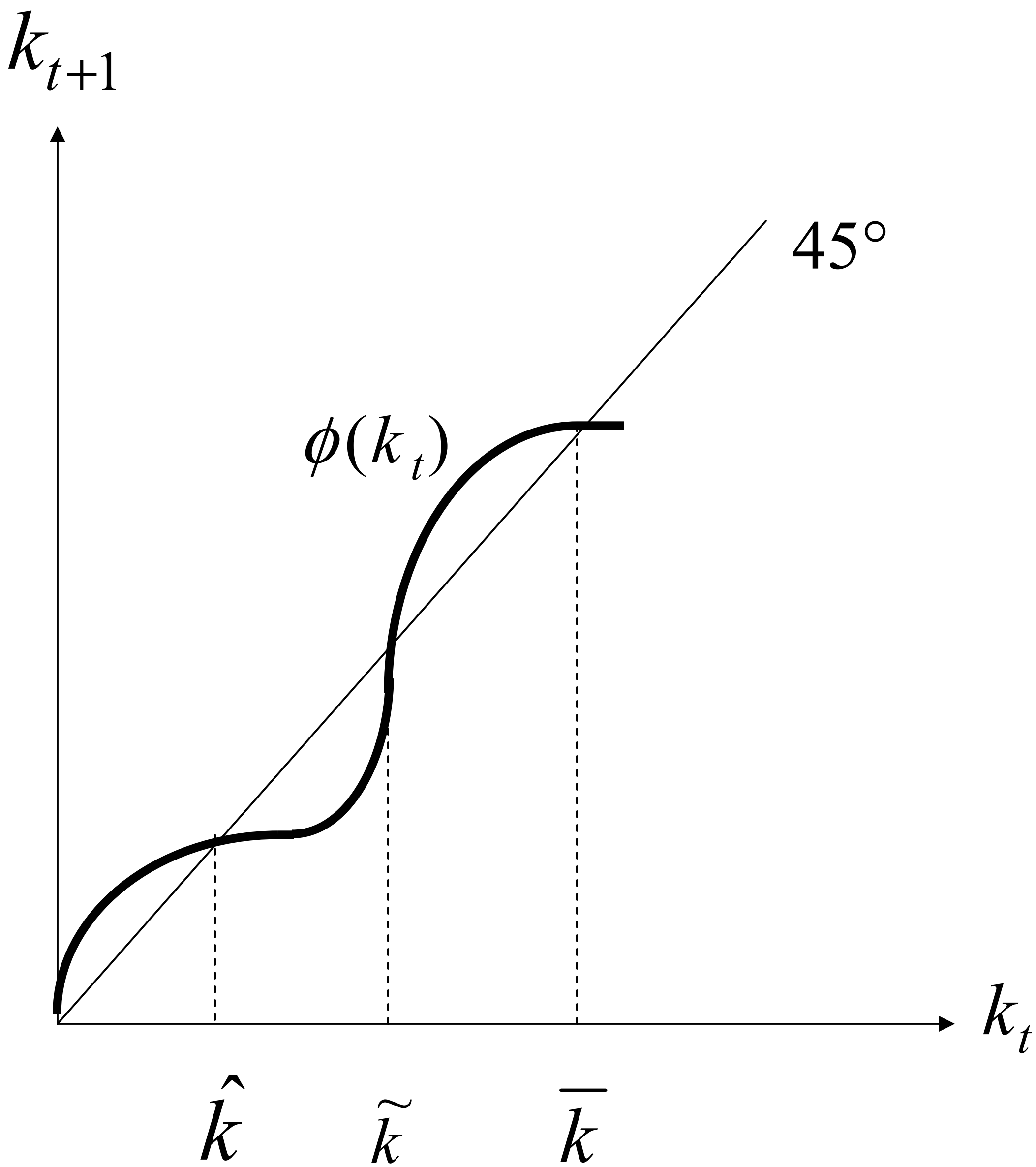
The sign of $\phi''(k_t)$ depends on the *third derivative* of production function — about which we have not made any assumptions

Multiple steady states possible, even with neoclassical production function (see problem set)

See figure – which are stable, unstable?



k_{t+1}
$\phi(k_t)$
45°
k_t
Indeterminate region



k_{t+1}
45°
$\phi(k_t)$
k_t
$\hat{k}$
$\tilde{k}$
$\bar{k}$

The Ramsey Model

Infinitely lived dynasty

Parents care about children

n = children per adult

α = weight on each child's utility = intergenerational discount factor

$u(c_t)$ = adult's utility from own consumption, c_t

Utility of generation t:

$$V_t = u(c_t) + \alpha n V_{t+1} \tag{17}$$

k_{t+1} = capital bequest per child

Budget constraint:

$$nk_{t+1} = w_t + (1 + r_t)k_t - c_t \tag{18}$$

Capital income: each agent endowed with k_t by her parent; principal + interest = $(1 + r_t)k_t$

Labor income: each agent supplies (inelastically) one unit of labor, paid w_t

Solving the model

Looking for two-dimensional system of difference equations, like this:

$$\begin{aligned} c_{t+1} &= g(c_t, k_t) \\ k_{t+1} &= h(c_t, k_t) \end{aligned} \tag{19}$$

Equation for k_{t+1} given by budget constraint in (18)

Need to find equation for c_{t+1} – given by the so-called Euler Equation

Derived from optimal intertemporal choice for consumption

Two approaches: the Bellman Equation and the Hamiltonian

I. *The Bellman Equation*

Define the value function $V(\cdot)$ from:

$$V(k_t) = \max_{k_{t+1} \geq 0} \left\{ \begin{array}{c} u(w_t + (1 + r_t)k_t - nk_{t+1}) \\ +\alpha n V(k_{t+1}) \end{array} \right\} \tag{20}$$

Note: value function is time-independent

FYI: existence of value function requires certain conditions to hold – see Contraction Mapping Theorem

FOC with respect to k_{t+1}:

$$-u'(c_t)n + \alpha n V'(k_{t+1}) = 0 \tag{21}$$

Find $V'(k_{t+1})$

Let k^*_{t+1} denote optimal k_{t+1}

Must be a function of k_t

From definition of $V(k_t)$ in (20)

$$\begin{aligned} V(k_t) = u(w_t + (1 + r_t)k_t - nk^*_{t+1}) \\ + \alpha n V(k^*_{t+1}) \end{aligned} \tag{22}$$

Differentiate with respect to k_t

$$V'(k_t) = u'(w_t + (1 + r_t)k_t - nk^*_{t+1})(1 + r_t)$$

$$+\frac{\partial k^*_{t+1}}{\partial k_t} \underbrace{\left\{-nu'(c_t) + \alpha n V'(k^*_{t+1})\right\}}_{=0 \text{ from FOC (21)}}$$

$$= u'(c_t)(1 + r_t) \tag{23}$$

This is the so-called *Envelope Theorem*

Note that (23) must hold for arbitrary t:

$$V'(k_{t+1}) = u'(c_{t+1})(1 + r_{t+1}) \tag{24}$$

Together (21) and (24) give the *Euler Equation:*

$$\frac{u'(c_t)}{u'(c_{t+1})} = \alpha(1 + r_{t+1}) \tag{25}$$

II. *The Hamiltonian*

Forward V_t in (17) to V_{t+1}; insert back into V_t

$$\begin{aligned} V_t &= u(c_t) + \alpha n V_{t+1} \\ &= u(c_t) + \alpha n \left[u(c_{t+1}) + \alpha n V_{t+2} \right] \\ &= ... = \textstyle\sum_{i=0}^{\infty} (\alpha n)^i u(c_{t+i}) \end{aligned} \tag{26}$$

Let $t = 0$ and rename index variable t

$$V_0 = \sum_{t=0}^{\infty} (\alpha n)^t u(c_t) \tag{27}$$

Rewrite budget constraint:

$$k_{t+1} = \frac{1}{n} \left[w_t + (1 + r_t) k_t - c_t \right] \tag{28}$$

Hamiltonian:

$$\begin{aligned} H(c_t, k_t, \lambda_{t+1}) &= (\alpha n)^t u(c_t) \\ &+ \lambda_{t+1} \left\{ \tfrac{1}{n} \left[w_t + (1 + r_t) k_t - c_t \right] \right\} \end{aligned} \tag{29}$$

Intuition: λ_{t+1} is the present value in utility terms of capital in period $t+1$ (the shadow price of k_{t+1}).

Optimality conditions:

$$\frac{\partial H(c_t, k_t, \lambda_{t+1})}{\partial c_t} = (\alpha n)^t u'(c_t) - \lambda_{t+1}/n = 0 \quad (30)$$

$$\frac{\partial H(c_t, k_t, \lambda_{t+1})}{\partial k_t} = \lambda_{t+1} \left(\frac{1 + r_t}{n} \right) = \lambda_t \qquad (31)$$

Deriving Euler Equation:

from (30): $(\alpha n)^t u'(c_t) = \lambda_{t+1}/n$

from (31): $(\alpha n)^t u'(c_t) n \left(\frac{1+r_t}{n}\right) = \lambda_t$

forward and divide by n: $(\alpha n)^{t+1} u'(c_{t+1}) \left(\frac{1+r_{t+1}}{n}\right) = \lambda_{t+1}/n$

back into first equation:

$$(\alpha n)^t u'(c_t) = (\alpha n)^{t+1} u'(c_{t+1}) \left(\frac{1+r_{t+1}}{n}\right) \quad (32)$$

which gives the Euler Equation:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \alpha(1 + r_{t+1}) \quad (33)$$

Dynamical system for the Ramsey Model

Recall: looking for $c_{t+1} = g(c_t, k_t)$ and $k_{t+1} = h(c_t, k_t)$

Budget constraint gives $k_{t+1} = h(c_t, k_t)$

Euler Equation and $r_{t+1} = f'(k_{t+1}) - \delta = f'(h(c_t, k_t)) - \delta$ gives $c_{t+1} = g(c_t, k_t)$

Example: Cobb-Douglas production ($y = k^\rho$), logarithmic utility [$u(c) = \ln(c)$], and full depreciation ($\delta = 1$)

$$k_{t+1} = \frac{1}{n}\left[k_t^\rho - c_t\right]$$

$$c_{t+1} = \alpha\rho \underbrace{\left(\frac{1}{n}\right)^{\rho-1}\left[k_t^\rho - c_t\right]^{\rho-1}}_{k_{t+1}^{\rho-1}} c_t \tag{34}$$

Analyzing the dynamics: the phase diagram

Illustration of a 2-dimensional dynamical system

Idea: see how c_t and k_t evolve over time at different positions in the (c_t, k_t)-space

Here: stick to parametric example in (34)

To find $(\Delta c_t = 0)$-locus, set $\Delta c_t = c_{t+1} - c_t = 0$:

$$(\Delta c_t = 0)\text{-locus:} \quad c_t = k_t^{\rho} - n(\alpha\rho)^{\frac{1}{1-\rho}} \tag{35}$$

To find $(\Delta k_t = 0)$-locus, set $\Delta k_t = k_{t+1} - k_t = 0$:

$$(\Delta k_t = 0)\text{-locus:} \quad c_t = k_t^{\rho} - nk_t \tag{36}$$

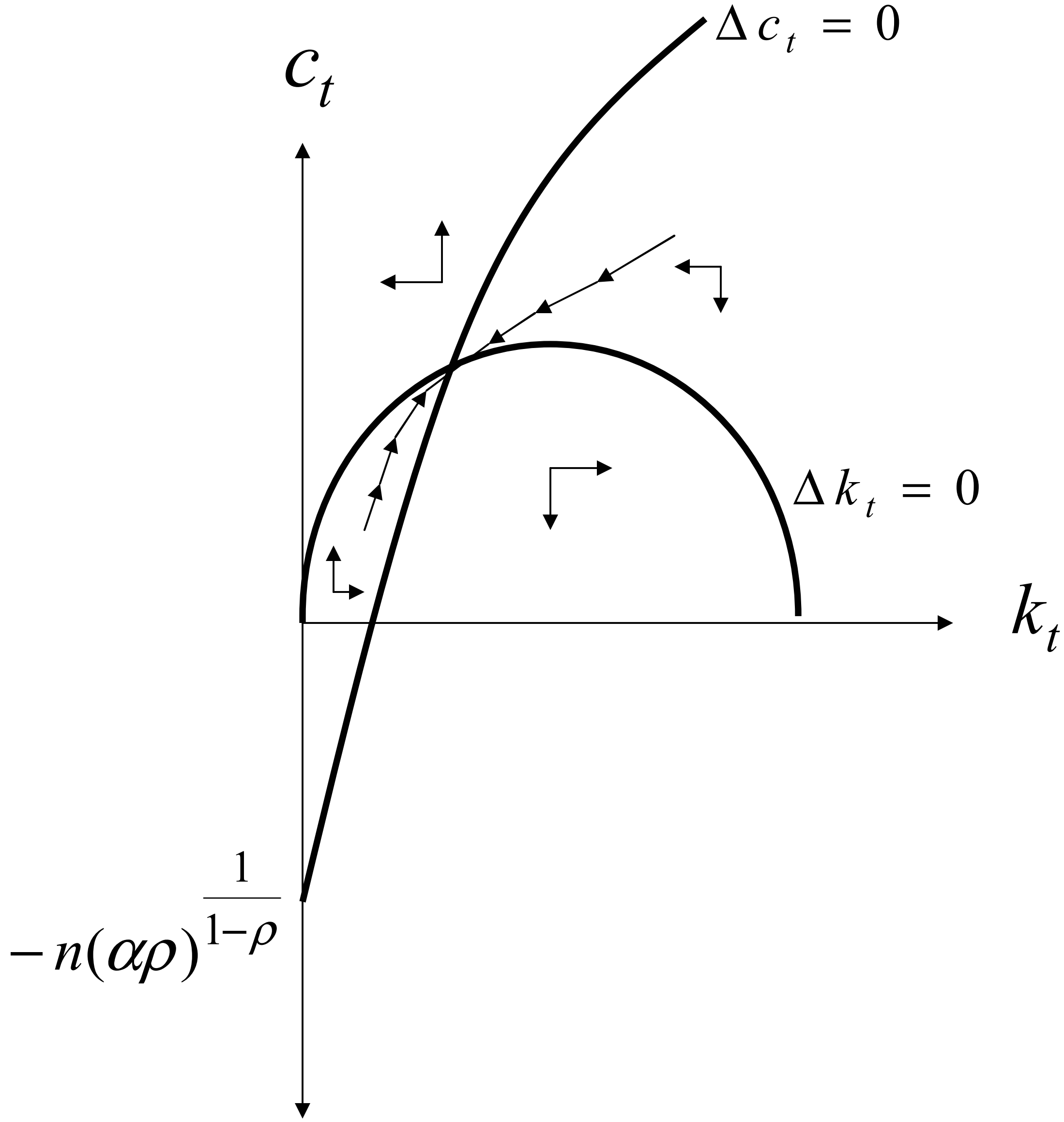
When $c_t > k_t^\rho - n(\alpha\rho)^{\frac{1}{1-\rho}}$: c_t is increasing over time

When $c_t < k_t^\rho - n(\alpha\rho)^{\frac{1}{1-\rho}}$: c_t is decreasing over time

When $c_t < k_t^\rho - nk_t$: k_t is increasing over time

When $c_t > k_t^\rho - nk_t$: k_t is decreasing over time

Exercise:
(a) Show that the diagram is drawn under the assumption $\alpha n < 1$.
(b) Given the utility function in (27), why must we assume $\alpha n < 1$?



c_t
$\Delta c_t = 0$
$\Delta k_t = 0$
k_t
$-n(\alpha\rho)^{\frac{1}{1-\rho}}$

Associated with each starting value (c_0, k_0) the dynamical system assigns a (unique) trajectory

To ensure that we have selected an optimal and feasible trajectory we impose a so-called *Tranversality Condition.* This essentially amounts to being on a trajectory leading to the steady state; called the *saddle path*

- Trajectories on which k_t becomes negative are not feasible.

- Trajectories on which c_t goes to zero are not optimal because Inada condition on utility function says $\lim_{c \to 0} u'(c) = \infty$. Since k_t is positive and finite, so is the interest rate: Euler Equation cannot hold.

Given k_0, there is a unique c_0 on the saddle path

Sustained exogenous growth

So far: k_t converges to $\overline{k}$, and y_t to $\overline{y} = f(\overline{k})$

Per-capita income stops growing in steady state: no *sustained* growth

Easy to fix; rename variables: $L_t = A_t \widetilde{L}_t$

$A_t =$ efficiency level of each worker

$\widetilde{L}_t =$ population (i.e., what was denoted L_t before)

Growth rates:

$$\begin{aligned} A_{t+1} &= (1+g)A_t \\ \widetilde{L}_{t+1} &= (1+\widetilde{n})\widetilde{L}_t \\ L_{t+1} &= (1+\widetilde{n})(1+g)L_t \equiv (1+n)L_t \end{aligned} \tag{37a}$$

Solving model (Ramsey or Solow) same as before; same dynamics for $k_t = K_t/L_t$

News: lower-case variables now denote levels *per effective worker:* $y_t = Y_t/L_t = Y_t/(A_t\widetilde{L}_t)$

In steady state with non-growing income per effective worker ($y_t = \overline{y}$) income per worker, $Y_t/\widetilde{L}_t = A_t\overline{y}$, grows at rate g

Shortcoming: sustained growth is *exogenous*, not explained in the model

Sustained endogenous growth

What stops per-capita income from growing in previous settings?

Answer: the assumption that $\lim_{k \to \infty} f'(k) = 0$

Instead assume that

$$\lim_{k \to \infty} f'(k) = A > \frac{\delta + n}{s} \tag{38}$$

Endogenous growth in a Solow setting

Recall:

$$\begin{aligned} \lim_{k_t \to \infty} \phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to \infty} f'(k_t) \\ &= \tfrac{1-\delta}{1+n} + \tfrac{sA}{1+n} > 1 \end{aligned} \tag{39}$$

Define the growth rate of k_t as

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} \tag{40}$$

The "steady state" growth rate is: $\overline{\gamma} = \lim_{k_t \to \infty} \gamma_t$

Called a balanced growth path

Use (39):

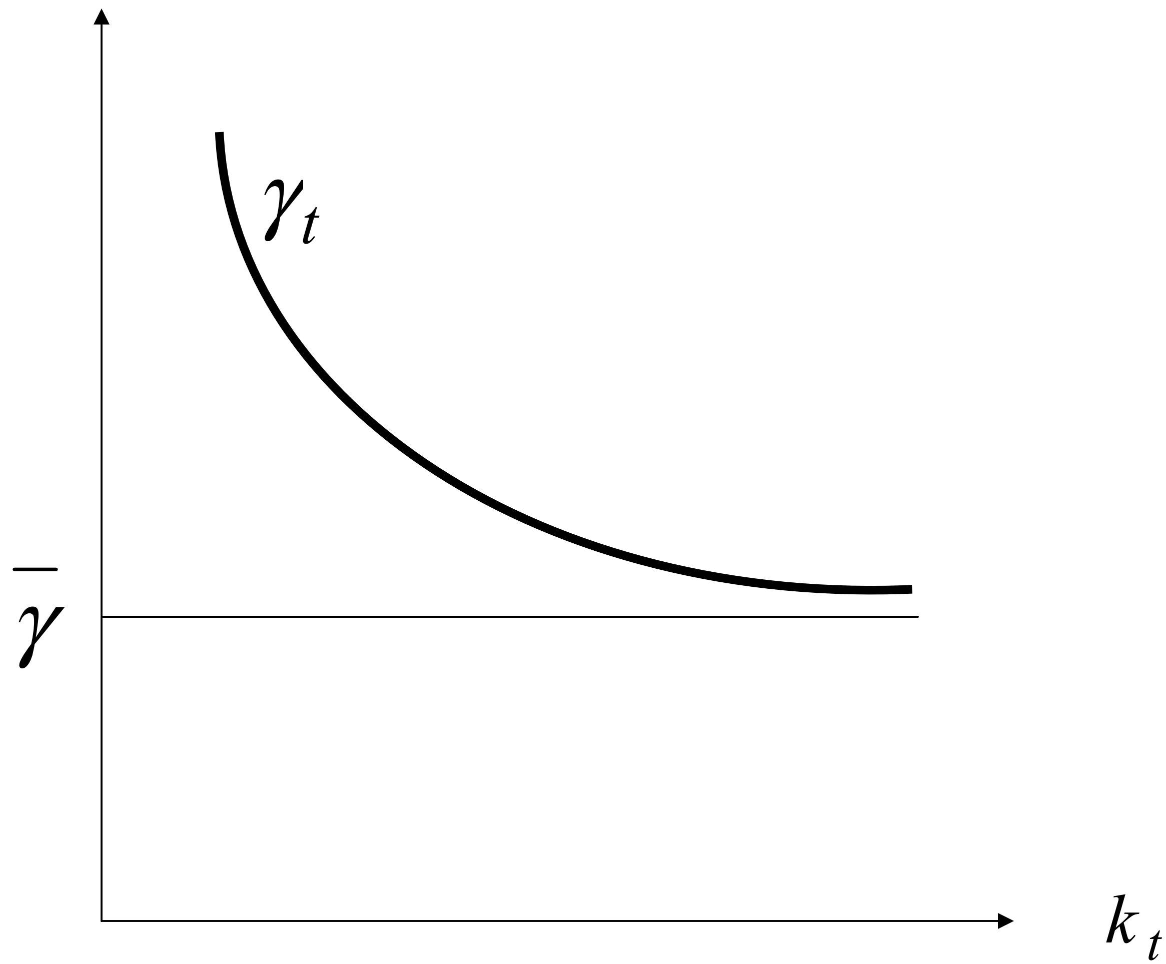
$$1 + \overline{\gamma} = \lim_{k_t \to \infty} \frac{k_{t+1}}{k_t} = \lim_{k_t \to \infty} \frac{\phi(k_t)}{k_t}$$
$$= \lim_{k_t \to \infty} \frac{\phi'(k_t)}{1} = \left\{ \frac{1-\delta}{1+n} + \frac{sA}{1+n} \right\} > 1 \quad (41)$$

from l'Hôpital's Rule and the assumption about A in (38)

Rewriting:

$$\overline{\gamma} = \frac{sA - (\delta + n)}{1 + n} \quad (42)$$

Higher saving generates faster *growth* (before: only level effects)



γ_t
$\bar{\gamma}$
k_t

Endogenous growth in a Ramsey setting

Consider example logarithmic utility and full depreciation. Using Euler Equation in (33):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \alpha f'(k_{t+1}) \tag{43}$$

Again: assume $\lim_{k\to\infty} f'(k) = A$

Define the rate of saving as

$$s_t = 1 - \frac{c_t}{f(k_t)} \tag{44}$$

On a balanced growth path s_t must be between 0 and 1; denote it $\overline{s} \in (0, 1)$

On a balanced growth path the consumption-capital ratio becomes:

$$\lim_{k_t\to\infty} \frac{c_t}{k_t} = (1 - \overline{s}) \lim_{k_t\to\infty} \frac{f(k_t)}{k_t} = (1 - \overline{s})A \tag{45}$$

implying that on BGP consumption and capital grow at same rate: $c_{t+1}/c_t = k_{t+1}/k_t \equiv 1 + \overline{\gamma}$

Use (43):

$$\overline{\gamma} = \alpha A - 1 \tag{46}$$

Higher weight on the future (higher α) generates faster *growth* (before: only level effects). Similar to growth effects in Solow model from changes in s

Solving for steady state rate of saving: use (45), (46), and budget constraint:

$$\underbrace{\frac{k_{t+1}}{k_t}}_{\to 1+\overline{\gamma}=\alpha A} = \frac{1}{n}\left[\underbrace{\frac{f(k_t)}{k_t}}_{\to A} - \underbrace{\frac{c_t}{k_t}}_{\to(1-\overline{s})A}\right] \tag{47}$$

gives $\overline{s} = \alpha n$

The AK model

Simple endogenous growth model: Cobb-Douglas with capital share $= 1$

$$F(K, L) = AK, \text{ so that } f(k) = Ak, \text{ for all } k$$

Called the AK model

In Solow setting:

$$\frac{k_{t+1}}{k_t} = \frac{sA + 1 - \delta}{1 + n} > 1 \tag{48}$$

Similar to (41); now $f(k_t) = Ak_t$ in each period not only in the limit

Endogenous growth model with human capital

Let there be two accumulable factors: physical capital (K), and human capital (H)

Simple Cobb-Douglas example; capital share $= \alpha$.

$$F(K, H) = K^{\alpha} H^{1-\alpha} \tag{49}$$

Let r_H and r_K denote the return to human and physical capital, respectively

Same depreciation rate, δ

Profit maximization:

$$\begin{aligned} r_H &= (1-\alpha)\left(\frac{K}{H}\right)^{\alpha} - \delta \\ r_K &= \alpha\left(\frac{K}{H}\right)^{\alpha-1} - \delta \end{aligned} \tag{50}$$

In equilibrium both assets must earn same return: $r_H = r_K = r$

Gives constant physical-to-human capital ratio:

$$H = \left(\frac{1-\alpha}{\alpha}\right) K \tag{51}$$

Substitute back into production function:

$$\begin{aligned} F(K,H) &= K^{\alpha}\left[\left(\tfrac{1-\alpha}{\alpha}\right)K\right]^{1-\alpha} \\ &= \left(\tfrac{1-\alpha}{\alpha}\right)^{1-\alpha} K \equiv BK \end{aligned} \tag{52}$$

Same structure as in AK model; now the “BK” model

Same endogenous growth results

Testing growth models against data

If interpreted literally neo-classical growth models predict countries grow faster the farther they are from steady state

That is: poor countries should grow faster than (and be catching up with) rich countries

Called *(absolute) convergence*; not consistent with data: see Barro (1997, Figure 1.1)

However: most economies differ from each other in exogenous parameters: e.g. n, s, and/or the production function (different Total Factor Productivity, TFP)

Must reinterpret convergence prediction

Conditional convergence: economy grows fast if its far below its "own" steady state

Some notation:

Initial income=y_0, steady state income=y^*

Linear approximation around steady state: growth proportional to $(y^* - y_0)$

Fast growth if y_0 small, or y^* large

Problem when testing theory: countries with low y_0 (i.e. developing countries) often have low y^*

Need to measure y^*

From model: y^* depends on e.g. n, s; so we can calibrate y^* to see how fast a country "should" grow according to model. Residual = TFP. Called growth accounting

But harder to measure production function: taxes, market distortions, rule of law, political freedom, terms of trade

Empirical approach: cross-country growth regressions

Idea: let each country on planet Earth be one "experiment," one *observation*

Growth model thought of as the data generating process

Data:

GDP/capita and more in the Summer and Heston's *World Penn Tables*

http://datacentre.chass.utoronto.ca/pwt/

Other data from other sources, e.g. Barro and Lee for schooling

In all: data for not more than some hundred countries

To get more observations (higher degrees of freedom) create a panel: each country measured over several periods (typically decades)

Why not one period = one year? Error terms not serially uncorrelated; want to measure growth, not cycles

The regression equation

See Barro (1997)

Left-hand side: annual growth rate across different countries and periods (e.g. 1960 -70, 1970-80, and 1980-90)

Let $Y_{i,t}$ = GDP/capita in country i period t
$y_{i,t} = \ln(Y_{i,t})$
$g_{i,t}$ = annual % growth rate
n = number of years

$$\frac{y_{i,t+1} - y_{i,t}}{n} = \ln(1 + g_{i,t}) \approx g_{i,t}$$

Approximation close if $g_{i,t}$ small (which it is, usually a couple of %)

Right-hand side: intercept, initial income, things controlling for y^* – call them $x_{i,t}$, and an error term

Regression equation:

$$g_{i,t} = \beta_0 + \beta_1 y_{i,t} + \beta_2 x_{i,t} + \varepsilon_i$$

Testing for conditional convergence*:* poor countries grow faster than rich if you control for the steady state they converge to

Convergence means: $\beta_1 < 0$

Barro finds $\widehat{\beta}_1$ to be highly significant

More generally:
If *not* controlling for anything else (leaving out the x_i's): $\widehat{\beta}_1$ typically insignificant (and/or wrong sign)
If controlling for things that proxy for y^* (letting the x_i's enter on the right-hand side): $\widehat{\beta}_1$ highly significant

Conclusion: *conditional* convergence, but no *absolute* convergence

Speed of convergence

$\widehat{\beta}_1 = -.025$; means gap closes by 2.5% per year

Gap T years ahead $= (1 - .025)^T$ of gap today.

How long time to half the gap? Set $(1-.025)^T = 1/2$, solving for T, gives 27 years

Galton's fallacy

If poor countries grow faster than rich, does that imply a negative time trend in inequality? Not necessarily; called Galton's fallacy

Assume $-1 < \beta_1 < 0$ in regression equation (holds in data) and disregard the $x_{i,t}$'s

Recall $g_{i,t} = (y_{i,t+1} - y_{i,t})/n$

$$y_{i,t+1} - y_{i,t} = n\beta_0 + n\beta_1 y_{i,t} + n\varepsilon_i$$

Say $\varepsilon_i \sim (0, \sigma_\varepsilon)$. Write period-$t+1$ variance in log GDP/cap, as function of that in period t:

$$V_{t+1} = \underbrace{(1 + n\beta_1)^2}_{<1} V_t + n^2\sigma_\varepsilon$$

Steady state variance: $V^* = \sigma_\varepsilon/[1 - (1 + n\beta_1)^2]$
Variance may increase or decrease, depending on where V_t is relative to V^* (draw 45-$^\circ$ diagram to see)

Other findings

Coefficient on schooling > 0 and significant (how about male vs. female schooling?)
Coefficient on fertility < 0 and significant
Coefficient on government consumption < 0 and significant
Coefficient on rule of law > 0 and significant
Coefficient on terms of trade > 0 and significant
Coefficient on regional dummies (Latin America, Africa) insignificant: other variables explain why they don't grow

Summary: more schooling, lower fertility, smaller government, and better rule of law (e.g. less corruption) all good for growth

Common mistakes when running cross-country growth regressions:

- Using total GDP instead of GDP/capita
  - Like comparing heights of Canadians and Swedes and summing up the height of all Swedes and all Canadians (standing-on-shoulders comparison)

- Using current prices instead of constant prices (not correcting for inflation)

- Series should be PPP adjusted; do not use current exchange rates

From World Penn Tables use (for example):

REAL GDP PER CAPITA (CONSTANT PRICE: LASPEYRES) (unit $ CONSTANT)

Discussion

New cross-country data (World Penn Tables) generated increased interest in growth models in the 80's

Useful insights, but also many problems, econometric and others (see Mankiw 1995)

Simultaneity: right-hand side variables not exogenous
Solution: find exogenous instruments, often lagged variables (cf Barro 1997); Acemoglu et al. (AER 2001) use European settler mortality rates

Multicollinearity: explanatory variables correlated with each other; Mankiw: "those countries that do things right do most things right, and those countries that do things wrong do most things wrong."

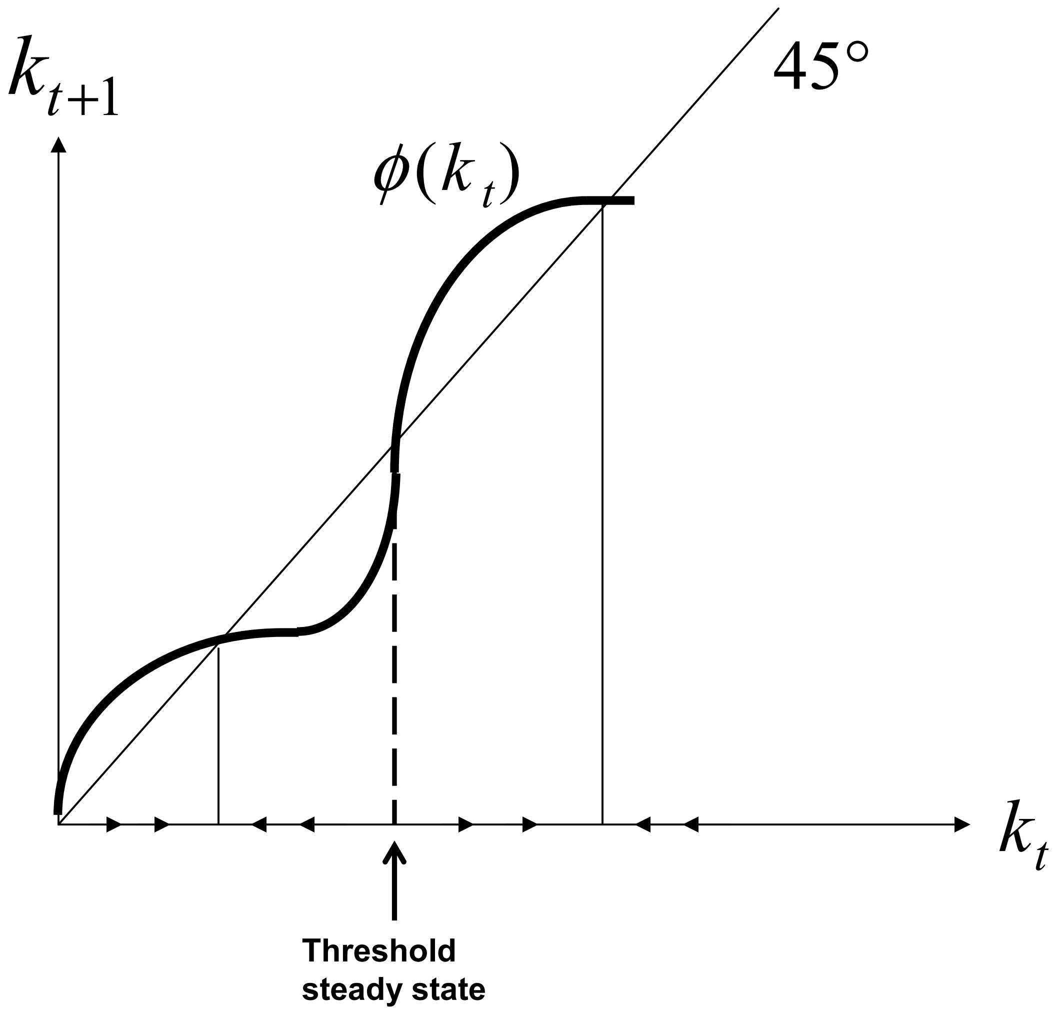
Degrees-of-Freedom: few observations, only about 100 countries
Solution: use panels; but that gives no new independent observations; also problem with business cycles vs. growth

Club convergence

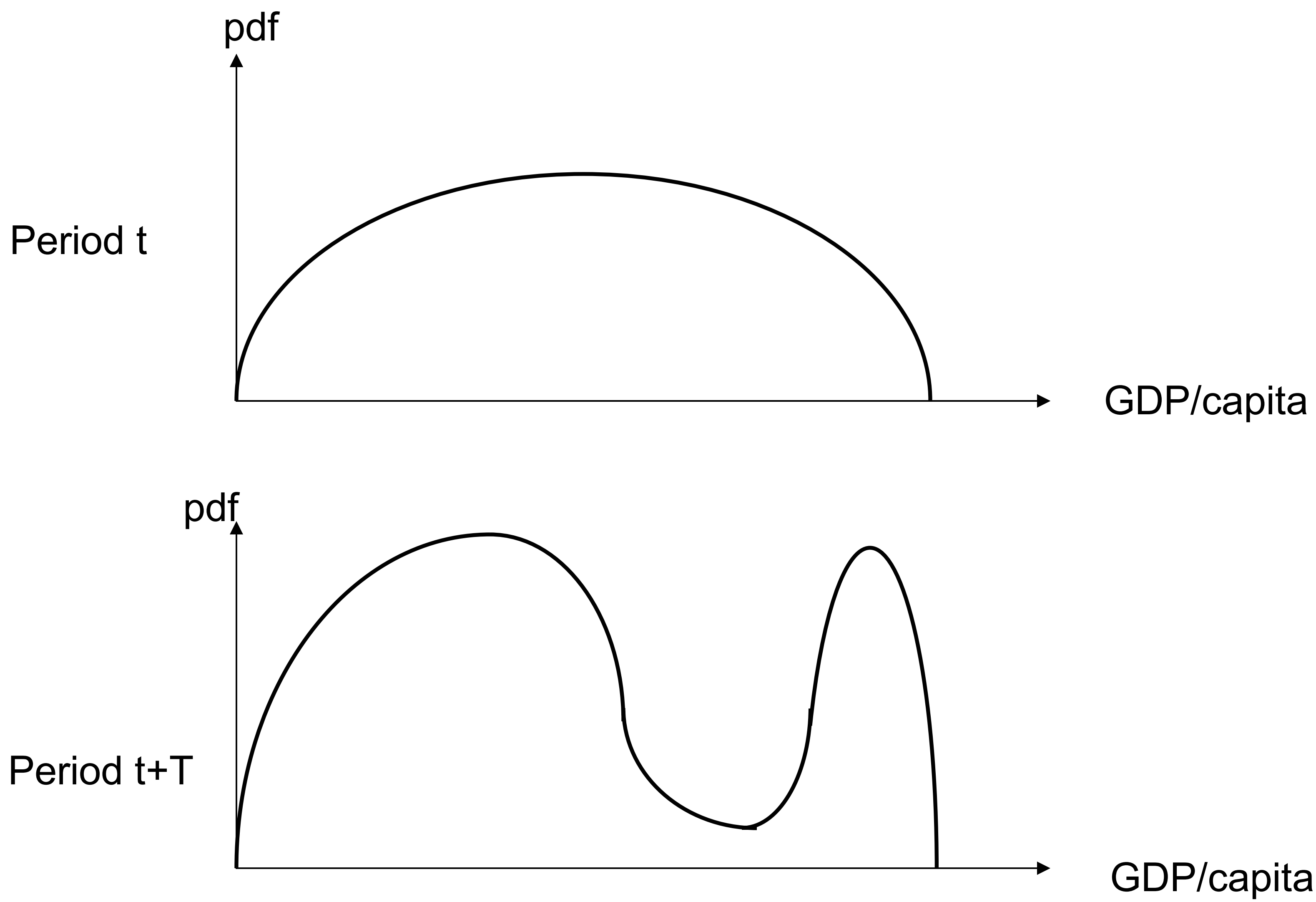
Consider setting with multiple steady states:
below some threshold: convergence to poverty trap
above that threshold: convergence to rich steady state

Implications for the distribution of per-capita income:
emerging twins peaks

Supported by data: Quah (1997)



k_{t+1}
45°
$\phi(k_t)$
k_t
Threshold
steady state


pdf
Period t
GDP/capita
pdf
Period t+T
GDP/capita

Taking longer perspective

Cross-country growth comparisons from1870 until today: Pritchett (1997)

1870 earliest year from when we have data; also “birth” date of many countries; data for 17 countries (Pritchett 1997, Table 1)

All rich today: “advanced capitalist” countries

Selectivity problem: data set consists of those which are rich today, and were rich 1870 (Europe), and those which are rich today, and were poor 1870 (Japan)

No data for countries that were rich 1870, but (relatively) poor today (Argentina), or countries which were poor in 1870, and poor today (India)

Convergence almost tautological: poor in 1870 must have grown faster since they’ve joined the “rich today” group

Prichett's approach: when 1870 data not available, set it at lower bound for per-capita income (subsistence level for people to survive)

Calculate various measures of dispersion of implied income distribution, and compare to the same measure today – which we know from data

Result: Table 2 in Pritchett

Conclusion: no convergence but "divergence, big time"

Unified approach

Explaining both divergence and convergence

Lucas' (2002) story: countries make random exogenous take-offs from stagnant steady state to sustained growth path

Long time ago: all countries at subsistence consumption; no gap
By 1870: some countries had started growing; others had not; gap starts increasing
Today: growth club more ahead of non-growers – but more countries in growing club

Countries which recently joined the growing club grow faster (due to convergence within the growing club)
Gap has increased so far, due to growers getting more ahead of non-growers
Future: as all join the growing club the income gap must start to decrease

Simulating time paths: Figures 1 - 3 in Lucas

World average growth rate first increases over time, as more and more countries start growing; poor countries grow faster then starts declining, as more rich countries and rich countries grow slower

Over time: humpshaped pattern

Standard deviation in per-capita income is increasing at first as the early growers take off; decreasing later as more countries start growing, catching up with leaders

Over time: humpshaped pattern

Summary: So far: Divergence, Big Time! Sooner or later: Convergence. More "optimistic" view than Pritchett

The distribution of individual incomes

So far: usually one country = one data point. What if we think of *individuals* as data points?

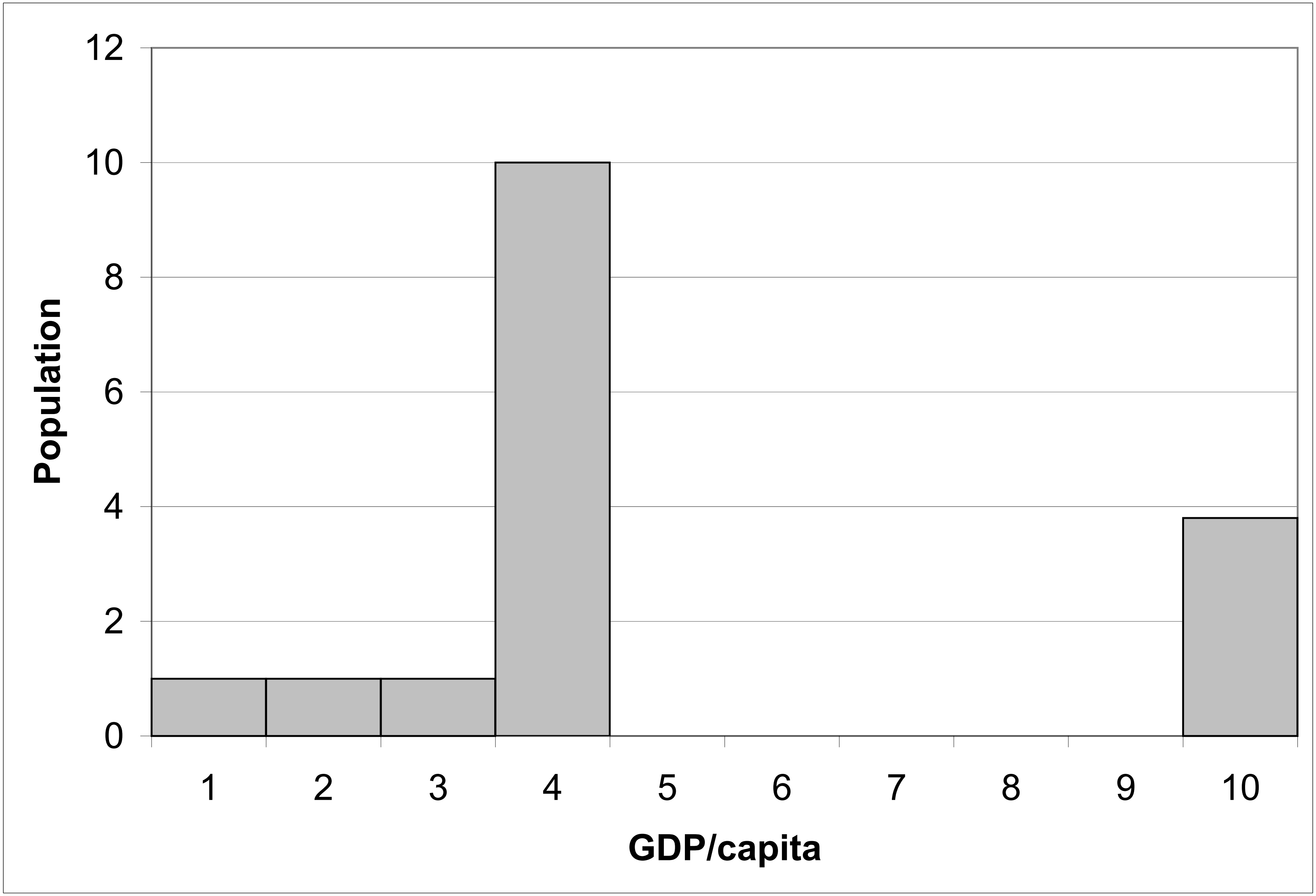
Consider hypothetical world distribution of income as below

3 poor countries with population = 1 each (Sub-Saharan Africa)
GDP/cap levels = 1, 2 and 3, resp.

1 semi poor country with population = 10 (China)
GDP/cap = 4

1 rich country with population = 3.8 (USA)
GDP/cap = 10

No income dispersion within each country



Population
GDP/capita
0
2
4
6
8
10
12
1
2
3
4
5
6
7
8
9
10

Suppose China grows from GDP/capita = 4 to 5?

If 1 country = 1 observation
Initial mean = (1+2+3+4+10)/5=4
China growing (=departing from mean) raises inequality

If each country's weight = population size: initial mean=5
China growing (approach mean) lowers inequality

Moreover: roughly this has happened

Still not perfect measure: all persons within country not identical; would like each *person* = one observation

Sala–Martin (2002) starts by estimating distributions of income within countries, then aggregates to one world income distribution.

Hard due to lack of data; requires some econometric fiddling. Here: focus on results

Sala's Figure 2 shows how distributions for different countries have evolved over time; most have grown, but not all (e.g. Nigeria); some have twin peaks, including the US

Figure 3 aggregates all distributions into one world distribution; in 1970 almost twin peaks, in 1990 much smoother; growth of China and India even out the hump

Figure 4 shows world distribution in one single diagram (as pdf and cdf)

Poverty rates (i.e., fraction living under \$1 and \$2) have fallen: see area to the left of threshold in pdf diagram

1998 first-order stochastically dominates 1970, meaning that at any level of GDP/cap, smaller mass lived below that level in 1998 than 1970 (however 1998 does not dominate 1990)

Figure 6: poverty head counts = # people living below \$1 and \$2 (not fraction), has also fallen

Figure 10 shows different measures of income inequality: all have fallen but not monotonically; e.g. variance in log income starts to increase in the 90's