Problems for the graduate M.A. course in Economic Growth
York University, Winter 2004

Problem Set 1 (due in class January 26th, 2004):
Solve problems 1 to 5 below.

Problem Set 2 (due in class February 9th, 2004):
Solve problems 6 to 10 below.

Problem Set 3, which we make the last one (due in class March 29th, 2004):
Solve problems 11 to 16 below.

(1) For this problem you may use a mini calculator.

(a) If an economy's per-capita income grew at a constant rate every year for 10 years and doubled in that period, what was the annual growth rate?

(b) Per-capita income of some economy A grows an annual rate of 3%, and that of another economy B at 2%. If B's per-capita income is initially double that of A, how many years does it take A to catch up with B.

(2) Use a 45°-diagram to illustrate the dynamics of the difference equation, $x_{t+1} = \phi(x_t)$, in the parametric cases below. Be careful to draw the graphs of $\phi(x_t)$ correctly.

(a) $\phi(x_t) = a + bx_t$, where $a > 0$ and $b \in (0, 1)$.
(b) $\phi(x_t) = a(x_t)^b$, where $a > 0$ and $b \in (0, 1)$.
(c) $\phi(x_t) = \max\{0, 2\sqrt{x_t} - a\}$, where $a \in (0, 1)$.
(d) $\phi(x_t) = ax_t/(b + x_t)$, where $a > b > 0$.
(e) $\phi(x_t) = ax_t - bx_t^2$, where $a > 1$ and $b > 0$.

(3) For each of the cases in **(2)**, find analytical expressions for all steady-state equilibria (including the trivial zero steady state). Also, check analytically (without using a diagram) if they are (locally) stable or unstable. Is any of the steady states oscillatory?

(4) Consider the difference equation: $x_{t+1} = 2x_t(2 - x_t)$. Here your answers from **(2)** and **(3)** (e) should be useful.

(a) Are there any stable steady states to this dynamical system?

(b) If we start off with $x_0 = 2$, what can we say about the whole time path for x_t? That is, find a sequence $\{x_t\}_{t=0}^{\infty}$ which solves this difference equation for the initial value $x_0 = 2$.

(c) Do the same as in (b) above, but now instead let $x_0 = 1$.

(d) Now assume that $x_0 \in (0, 2)$, but $x_0 \neq 1$. Use your favorite program for simple numerical calculations (like Excel) to generate a time path for x_t. Try a couple of different start values, and run the simulation for 20-30 periods. Generate a graph showing the time path for x_t.

(5) Consider the profit maximization problem faced by an atomistic firm, taking wages, w, and the real interest rate, r, as given: $\max\limits_{K,L} \pi(K, L)$, where

$$\pi(K, L) = F(K, L) - wL - (r + \delta)K. \tag{1}$$

This is a two-variable maximization problem so the second-order condition is a little more complicated compared to the single-variable case. To see this, rewrite it as a single variable maximization problem, by first defining optimal K as a function of L. That is, let

$$K(L) \equiv \arg\max_K \{\pi(K, L)\}$$

and

$$\pi(L) \equiv \pi\left(K(L), L\right).$$

Let $L^* \equiv \arg\max_L \{\pi(L)\}$, and $K^* \equiv K(L^*)$. We can now think of the profit maximization problem as a single-variable problem, $\max\limits_{L} \pi(L)$, with first-order condition (for a local maximum) $\pi'(L^*) = 0$, and second-order condition $\pi''(L^*) \leq 0$.

To denote partial derivatives of multiple-variable functions we use the notation $\frac{\partial F(K,L)}{\partial K} = F_K(K, L)$, $\frac{\partial^2 F(K,L)}{\partial K^2} = F_{KK}(K, L)$, etc.

(a) Find an expression for $K'(L)$ in terms of partial derivatives of $F\left(K(L), L\right)$.

(b) Express the second-order condition $\pi''(L^*) \leq 0$ in terms of partial derivatives of $F(K^*, L^*)$. Is $F_{KK}(K^*, L^*) < 0$ and $F_{LL}(K^*, L^*) < 0$ sufficient for the second-order condition to hold?

(6) (Azariadis and Drazen 1990). Introduce a *threshold externality* into the Solow model, by letting total factor productivity depend on k_t:

$$f(k_t) = A(k_t)k_t^{\alpha},$$

where

$$A(k) = \begin{cases} \overline{A} & \text{if } k \geq \widehat{k} \\ \underline{A} & \text{if } k < \widehat{k} \end{cases},$$

and $\overline{A} > \underline{A}$ and $\widehat{k} > 0$.

(a) Draw the graph of $f(k_t)$.

(b) This production function exhibits a so-called *non-convexity*. In what sense?

(c) Find conditions on exogenous parameters (such as $\overline{A}$, $\underline{A}$, and $\widehat{k}$) under which the dynamics display multiplicity of steady states.

(7) Galor (1996) considers a version of the Solow model where saving out of labor income, s^w, differs from that of capital income, s^r. This gives the following dynamic equation for k_t:

$$k_{t+1} = \frac{s^w\{f(k_t) - f'(k_t)k_t\} + s^r f'(k_t)k_t + (1-\delta)k_t}{1+n}.$$

[Here we follow Galor's formulation and let capital income be given by $f'(k)k$, rather than $\{f'(k) + 1 - \delta\}k$.] We can write the (net) growth rate of k_t, as

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \xi(k_t) - (\frac{n+\delta}{1+n}),$$

where

$$\xi(k_t) = \frac{s^w\left[\frac{f(k_t)}{k_t}\right] + (s^r - s^w)f'(k_t)}{(1+n)}.$$

(a) Verify that any steady state must satisfy Galor's Eq. (3), i.e., $s^w f(\overline{k}) + (s^r - s^w)f'(\overline{k})\overline{k} = (n+\delta)\overline{k}$.

(b) Show that $\lim_{k \to 0} \xi(k) = \infty$ and $\lim_{k \to \infty} \xi(k) = 0$. [The Inada conditions on $f(k)$ are assumed to be satisfied.]

(c) Assume that $\xi'(k) > 0$ for some $k > 0$. Use the result in (d) above to demonstrate that – for some n and δ – the model generates multiple steady states. Illustrate in a diagram, and show which steady states are stable.

(d) Show that $\xi'(k) > 0$ is equivalent to $(s^r - s^w)f''(k) - s^w\left[\frac{w(k)}{k^2}\right] > 0$, where $w(k)$ denotes the wage rate. What is a necessary condition on s^w and s^r for there to be multiple steady states?

(8) This problem shows that the Diamond model can generate multiple steady states, and refers to the example in Azariadis (1994, pp. 203-204). Let production exhibit constant elasticity of substitution (CES), i.e.,

$$F(K, L) = A[aK^{-\rho} + (1-a)L^{-\rho}]^{-\frac{1}{\rho}},$$

where $1/(1+\rho)$ is the elasticity for substitution between capital and labor. Assume that this elasticity is less than one, i.e., $\rho > 0$. [More generally it would hold that $\rho \in (-1, \infty)$.]

Assume logarithmic utility, so that saving is some constant fraction of labor income. Show that the dynamics would be characterized by multiple steady states for some level of total factor productivity A.

(9) Consider the Barro and Becker (1989) model.

(a) Rewrite the budget constraint for C_t [see (59) in the notes] in terms of N_t, K_t, N_{t+1}, K_{t+1}, w_t, β_t, and r_t.

(b) Using the notation $V_t = N_t^{1-\varepsilon-\sigma}C_t^{\sigma} + \alpha V_{t+1}$ (see the notes) formulate a Bellman equation where the value function is given by $V(K_t, N_t)$.

(c) From the first-order condition for K_{t+1}, and Envelope, show that: $(C_t/C_{t+1})^{\sigma-1} = \alpha n_t^{1-\varepsilon-\sigma}(1 + r_{t+1})$.

(d) From the expression in (c), derive the Euler equation for $(c_t/c_{t+1})^{\sigma-1}$ [see (60) in the notes].

(e) From the first-order condition for N_{t+1}, Envelope, and the expression in (c), derive (65) in the notes:

$$c_{t+1} = \left(\frac{\sigma}{1-\sigma-\varepsilon}\right)[\beta_t(1 + r_{t+1}) - w_{t+1}]$$

(10) Consider the function determining unconstrained fertility in the Galor and Weil (1996) model:

$$\psi(k_t) = \gamma\left[2 + \frac{b[2-\psi(k_t)]^{\alpha}}{(1-\alpha)k_t^{\alpha}}\right].$$

Show that $\psi'(k_t) < 0$.

(11) Consider the Kremer model, presented in class. Period-t output is given by

$$Y_t = A_t L^{1-\alpha} P_t^{\alpha}.$$

Technology evolves according to

$$A_{t+1} = B A_t^{1-\beta} P_t^{\beta},$$

and (here's the news) population evolves as follows:

$$P_{t+1} = n\left(\frac{Y_t}{P_t}\right) P_t,$$

where $n(\cdot)$ gives fertility as a function of per-capita income, Y_t/P_t. We let $\overline{y}$ be per-capita income at which population is constant.

(a) Find a parametric example for $n(\cdot)$, which contains $\overline{y}$, and which satisfies these three conditions: (i) $n'(Y_t/P_t) > 0$; (ii) $n(\overline{y}) = 1$; and (iii) is isoelastic, meaning $\ln[n(x)]$ is linear (or affine) in $\ln x$.

(b) Use your answer in (a) to write a dynamical system for A_t and P_t. That is, find an equation for A_{t+1} in terms of A_t and P_t and exogenous variables, and an equation for P_{t+1} in terms of A_t and P_t and exogenous variables.

(c) Draw a phase diagram with A_t on the vertical axis, and P_t on the horizontal axis. Draw the loci along which $\Delta A_t = 0$, and $\Delta P_t = 0$, and show the dynamic paths for A_t and P_t for different start values.

Hint: If you got the phase diagram right it should have a "threshold curve," which looks like a saddle path. Economies starting off below this will vanish (A_t and P_t will go to zero); and economies starting off above it will see A_t and P_t growing without bounds. Since these is no transversality condition imposed there is no reason to believe that any economy would position itself on the saddle path.

(12) This question refers to Diamond's "Guns, Germs, and Steel." What is the difference between a proximate and an ultimate explanation?

(13) Consider the model(s) in Lucas (2002, Ch. 5, Sec. 3), and the case with logarithmic utility. When there are no property rights to land, we saw that we could write the first-order condition for n_t as

$$(1-\beta)\frac{k}{c_t} = \beta\gamma\frac{1}{n_t}.$$

(a) Use this first-order condition and the budget constraint to find an expression for n_t as a function of $f(x_t)$ and exogenous parameters.

(b) Let $f(x_t) = Ax_t^{\alpha}$, and recall that $x_{t+1} = x_t/n_t$. Derive a difference equation for x_t and illustrate the dynamics and the steady state in a 45°-diagram. Is the steady state stable?

(c) How does the steady state level of x_t depend on γ, A, and k? What is the intuition?

Consider next the case where agents have property rights to land. With logarithmic utility and Cobb-Douglas production it can be seen that the value function takes the following form:

$$W(x_t) = \Gamma + \Phi \ln x_t,$$

where Γ and Φ are constants that depend on parameters of the model.

(d) Using this expression for the value function, show that optimal fertility becomes

$$n_t = \left(\frac{\Psi}{k}\right) Ax_t^{\alpha},$$

where Ψ depends on β, γ, and Φ.

(e) Substituting optimal n_t back into the max expression in the Bellman equation, we can write the value function, $W(x_t)$, as a linear function of $\ln x_t$. Having done so, show what the unknown Φ must be. You may check that it fits with Lucas' (2002, p. 133) result.

(14) Use your favorite program for simple numerical calculations (like Excel) to replicate the time paths for S_t and L_t in the simulations of the Brander-Taylor model shown in class. Use the numerical values in the notes.

(15) Consider the model of Galor and Weil (2000). Recall that optimal education, e_{t+1}, is given by

$$G(e_{t+1}, g_{t+1}) = (\tau + e_{t+1})\, h_e(e_{t+1}, g_{t+1}) - h(e_{t+1}, g_{t+1}) = 0$$

if $e_{t+1} > 0$, and $e_{t+1} = 0$ if $G(0, g_{t+1}) < 0$.

Now consider the parametric example used in the notes:

$$h(e_{t+1}, g_{t+1}) = \frac{e_{t+1} + \rho\tau}{e_{t+1} + \rho\tau + g_{t+1}},$$

where $\rho \in (0, 1)$.

(a) Show that optimal education in this parametric case is given by:

$$e(g_{t+1}) = \max\left\{0, \sqrt{g_{t+1}\tau(1-\rho)} - \rho\tau\right\}$$

(Disregard the 0 argument in the max expression if you find that confusing; just derive $e(g_{t+1}) = \sqrt{g_{t+1}\tau(1-\rho)} - \rho\tau$ assuming that $e_{t+1} > 0$.)

(b) Find $\widehat{g}$, i.e., the level of g_{t+1} below which $e(g_{t+1}) = 0$.

(c) Find a parametric expression for $h(g_{t+1}) \equiv h(e(g_{t+1}), g_{t+1})$. In words, this is the level of human capital in period $t+1$ as a function technological progress from period t to $t+1$, taking into account both the erosion effect from technological progress and parents' optimal response to it. The answer should look like this:

$$h(g_{t+1}) = \begin{cases} \text{something containing } \rho\tau \text{ and } g_{t+1} & \text{if } g_{t+1} \leq \widehat{g} \\ \text{something containing } \sqrt{\tau(1-\rho)} \text{ and } \sqrt{g_{t+1}} & \text{if } g_{t+1} \geq \widehat{g} \end{cases}$$

where $\widehat{g}$ is the solution you found in (b).

Next, as in the notes, let technological progress take this functional form:

$$g_{t+1} = g(e_t, L) = (e_t + \rho\tau)a(L)$$

where $a'(L) > 0$ and $\lim_{L\to\infty} a(L) \equiv a^* \in (0, \infty)$.

(d) If $a(L)$ is large enough there exists a steady state $(\overline{e}, \overline{g})$ at which $\overline{e} > 0$. Find parametric expressions for $\overline{e}$ and $\overline{g}$.

(e) Use your answer in (c) to find the corresponding level of human capital, $h(\overline{e}, \overline{g})$, as a function on $a(L)$.

(f) Use the expressions for $h(e_{t+1}, g_{t+1})$ and $g(e_t, L)$ above to find an expression for human capital in a steady state where $e = 0$. That is, find $h(0, g(0, L))$. Does human capital in the steady state with zero education differ from the steady state with positive education?

(g) At what level of $a(L)$ does the economy move from the steady state with no education and slow technological change, to the one with positive education and faster technological change?

(16) Consider the model of Lagerlöf (2003).

(a) Using the functional form for $A(P_t)$ given in the paper and in the lecture notes, find

$$\lim_{P_t \to \infty} A(P_t).$$

(b) Show how to derive the expression for optimal education time:

$$h_t = \frac{1}{1-\delta}\left[v(\delta-\rho) - \frac{L}{A(P_t)(L+H_t)}\right].$$

(c) Use the above expression for h_t, and find what h_t approaches as both P_t and H_t go to infinity.

(d) Use your result in (c) to find what B_t approaches as P_t and H_t go to infinity.

(e) Use the expression for human capital accumulation,

$$H_{t+1} = A(P_t)\left[L + H_t\right](\rho v + h_t),$$

to derive an expression for the gross growth rate of human capital, H_{t+1}/H_t, in an economy where both P_t and H_t exhibit sustained growth (i.e., both approach infinity).

(f) In an economy where both H_t and P_t exhibit sustained growth, and H_t grows faster than P_t, the survival rate, T_t, goes to 1. Thus, the birth rate derived in (d) determines the gross growth rate of P_t. Derive a parametric condition for this type of growth path to exist.

Solutions to selected problems

(1) (a) The annual growth rate was about 7.18% (since $1.0718^{10} \approx 2$); (b) It takes $\ln(2)/\ln(\frac{1.03}{1.02}) \approx 71$ years.

(2)

(c) $\phi(x_t)$ coincides with the x_t-axis for $x_t \in [0, (a/2)^2]$. Thereafter the slope is positive and diminishing, intersecting the 45°-line twice.

(d) Diminishing slope; should be drawn so that $\phi(x_t)$ is bounded from above by a.

(e) Should be hump-shaped, peaking at $x_t = a/2b$.

(3)

(a) One unique steady state, $\overline{x} = a/(1-b)$. Stable since $\phi'(\overline{x}) = b \in (0,1)$.

(b) Two steady states: $\overline{x} = 0$, and $\overline{x} = a^{\frac{1}{1-b}}$. $\overline{x} = 0$ is unstable since $\lim_{x\to 0} \phi'(x) = \infty$. $\overline{x} = a^{\frac{1}{1-b}}$ is stable since

$$\phi'(\overline{x}) = ab\,(\overline{x})^{b-1} = ab\left(a^{\frac{1}{1-b}}\right)^{b-1} = b \in (0,1).$$

(c) There are three steady states. First $\overline{x} = 0$, which is stable because $\phi'(0) = 0$. The other two are given by $\overline{x} = 1 - a + 2\sqrt{\overline{x}} - 1$. Let $z \equiv \sqrt{\overline{x}}$; this gives $z^2 - 2z + 1 = (z-1)^2 = 1 - a$, which has roots $z = 1 + \sqrt{1-a}$ and $z = 1 - \sqrt{1-a}$. This gives the two remaining steady states as $\overline{x}^* = \{1 + \sqrt{1-a}\}^2 > 1$ and $\overline{x}^{**} = \{1 - \sqrt{1-a}\}^2 \in (0,1)$. Since $\phi'(x) = 1/\sqrt{x}$, $\overline{x}^*$ is stable and $\overline{x}^{**}$ unstable.

(d) There are two steady states. First $\overline{x} = 0$, which is unstable, since $\phi'(x) = \frac{ab}{(b+x)^2}$, which gives $\phi'(0) = a/b > 1$. The other steady state is given by $\overline{x} = a - b$, which is stable, since $\phi'(a-b) = \frac{ab}{(b+a-b)^2}, = b/a \in (0,1)$.

(e) There are two steady states. First $\overline{x} = 0$, which is unstable, since $\phi'(x) = a - 2bx$, so $\phi'(0) = a > 1$. The other is given by $\overline{x} = a\overline{x} - b\overline{x}^2$ or $1 = a - b\overline{x}$, or $\overline{x} = (a-1)/b$. To check stability, note that $\phi'(\frac{a-1}{b}) = a - 2(a-1) = 2 - a$. The steady state is stable if, and only if, $2 - a \in (-1, 1)$. The upper bound is OK since $a > 1$ is assumed. The lower bound requires that $a < 3$. The steady state will be oscillatory if $2 - a < 0$, i.e., if $a > 2$.

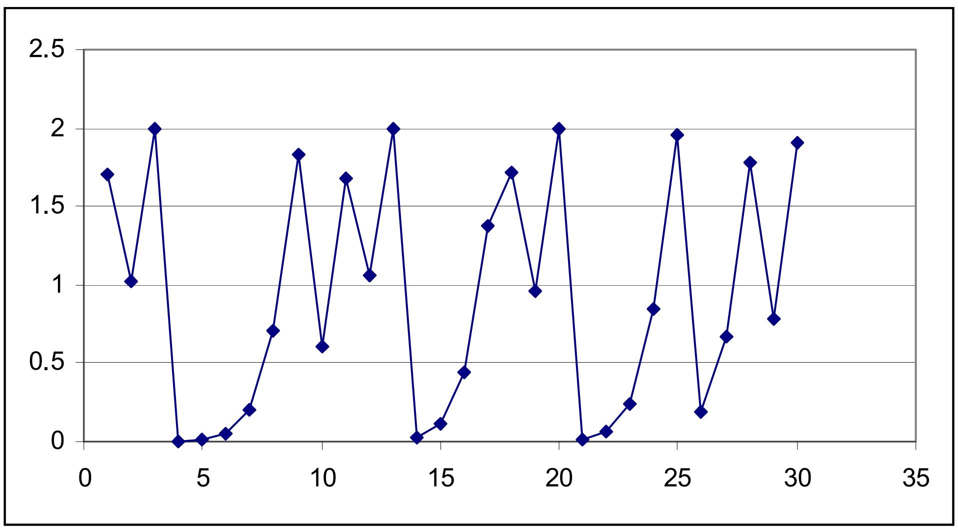
(4).

(a) No. See solution to (3) (e) above, setting $a = 4$ and $b = 2$.

(b) Starting off with $x_0 = 2$, x_t goes to the zero steady state in the next period and stays there forever. That is: $\{x\}_{t=0}^{\infty} = \{2, 0, 0, ...\}$.

(c) Starting with $x_0 = 1$ gives $x_1 = 2$ and the zero steady state forever after that. That is: $\{x\}_{t=0}^{\infty} = \{1, 2, 0, 0, ...\}$.

(d) I should have added that $x_0 \neq 1.5$, as well as $x_0 \in (0, 2)$, and $x_0 \neq 1$. (The unstable steady state is 1.5.) Then the time path for x_t should display chaotic behavior; it should look a bit like below.



(5)

(a) From the definition of $K(L)$ the first-order condition for K must hold at $K = K(L)$. So we can write $\pi_K(K(L), L) - (r + \delta) \equiv 0$. Thus, using the Implicit Function Theorem, we get

$$K'(L) = -\frac{\pi_{KL}(K(L), L)}{\pi_{KK}(K(L), L)} = -\frac{F_{KL}(K(L), L)}{F_{KK}(K(L), L)} \tag{2}$$

where the second equality comes from 1.

(b) First find $\pi'(L)$:

$$\pi'(L) = \pi_K(K(L), L)K'(L) + \pi_L(K(L), L)$$

Then find $\pi''(L)$:

$$\begin{aligned}\pi''(L) = \pi_K(K(L),L)K''(L) + \pi_{KL}(K(L),L)K'(L) + \pi_{KK}(K(L),L)\{K'(L)\}^2 \\ +\pi_{LL}(K(L),L) + \pi_{KL}(K(L),L)K'(L)\end{aligned} \tag{3}$$

where the first three terms are the derivative of $\pi_K(K(L),L)K'(L)$ with respect to L, and the last two terms are the derivative of $\pi_L(K(L),L)$ with respect to L.

Next, we evaluate $\pi''(L)$ at $L = L^*$. Since $K^* = K(L^*)$, this amounts to evaluating the terms on the right-hand side of (3) at K^* and L^*, and we know that $\pi_K(K^*,L^*) = 0$ by definition. Using (2) the remaining terms become:

$$\begin{aligned}\pi''(L^*) &= 2\pi_{KL}(K^*,L^*)K'(L^*) + \pi_{KK}(K^*,L^*)\{K'(L^*)\}^2 + \pi_{LL}(K^*,L^*) \\ &= -2\tfrac{[\pi_{KL}(K^*,L^*)]^2}{\pi_{KK}(K^*,L^*)} + \tfrac{[\pi_{KL}(K^*,L^*)]^2}{\pi_{KK}(K^*,L^*)} + \pi_{LL}(K^*,L^*) \\ &= \left(\tfrac{1}{\pi_{KK}(K^*,L^*)}\right)\left[\pi_{LL}(K^*,L^*)\pi_{KK}(K^*,L^*) - [\pi_{KL}(K^*,L^*)]^2\right] \\ &= \left(\tfrac{1}{F_{KK}(K^*,L^*)}\right)\left[F_{LL}(K^*,L^*)F_{KK}(K^*,L^*) - [F_{KL}(K^*,L^*)]^2\right].\end{aligned}$$

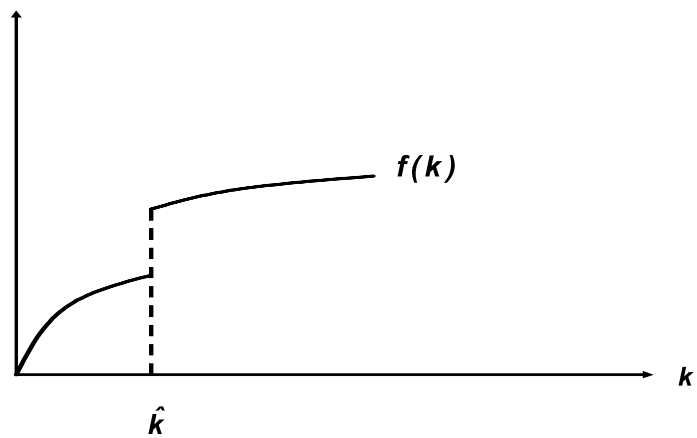
Since $F_{KK}(K^*,L^*) < 0$, for $\pi''(L^*) \leq 0$ to hold, the production function must be such that

$$F_{LL}(K,L)F_{KK}(K,L) - [F_{KL}(K,L)]^2 \geq 0$$

in optimum.

(6)

(a) The production function look roughly as below, making a ‘jump” at $k = \widehat{k}$.



(b) The production possibility set, $\left\{(k,y)\in\Re^2_+ : f(k)\geq y\right\}$, is non-convex, meaning a linear combination of two points in the set need not fall within the set.

(c) Calculating the two steady states associated with $\underline{A}$ and $\overline{A}$, $\widehat{k}$ must fall in between these. This gives:

$$\left(\frac{s\underline{A}}{n+\delta}\right)^{\frac{1}{1-\alpha}} < \widehat{k} \leq \left(\frac{s\overline{A}}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

(8) The intensive-form production function can be written as $f(k) = A[ak^{-\rho}+(1-a)]^{-\frac{1}{\rho}}$.

First we can see that: $f'(k) = aA^{-\rho}\left(\frac{f(k)}{k}\right)^{1+\rho}$.

Then some tedious algebra gives us: $w(k) = f(k) - f'(k)k = (1-a)A^{-\rho}[f(k)]^{1+\rho}$.

Implying that: $\lim_{k\to\infty} w(k) = (1-a)^{-\frac{1}{\rho}}A > 0$.

And: $\lim_{k\to 0} w'(k) = 0$.

In a standard Diamond setting it can be seen that k_{t+1} =const.$\times w(k)$, so given the behavior of $w(k)$ above we can get multiple steady states of we play around with A.

(9)

(a) $C_t = N_t w_t + (1+r_t)K_t - \beta_t N_{t+1} - K_{t+1}$. Note that $N_{t+1} = n_t N_t$.

(b) The Bellman equation becomes:

$$V(K_t, N_t) = \max_{N_{t+1}, K_{t+1}} \left[\begin{array}{c} N_t^{1-\varepsilon-\sigma} (N_t w_t + (1+r_t)K_t - \beta_t N_{t+1} - K_{t+1})^\sigma \\ +\alpha V(K_{t+1}, N_{t+1}) \end{array} \right]$$

(c) The first-order condition for K_{t+1} gives $N_t^{1-\varepsilon-\sigma}\sigma C_t^{\sigma-1} = \alpha V_K(K_{t+1}, N_{t+1})$. Using Envelope, we can write $V_K(K_t, N_t) = N_t^{1-\varepsilon-\sigma}\sigma C_t^{\sigma-1}(1+r_t)$ (plus zero terms). Forwarded one period, and using $N_{t+1} = n_t N_t$, this gives: $(C_t/C_{t+1})^{\sigma-1} = \alpha n_t^{1-\varepsilon-\sigma}(1+r_{t+1})$.

(d) Using $C_t = c_t N_t$, and $N_{t+1} = n_t N_t$, the Euler equation follows from the expression in (c): $(c_t/c_{t+1})^{\sigma-1} = \alpha n_t^{-\varepsilon}(1+r_{t+1})$.

(e) The first-order condition for N_{t+1} gives

$$N_t^{1-\varepsilon-\sigma}\sigma C_t^{\sigma-1}\beta_t = \alpha V_N(K_{t+1}, N_{t+1}). \tag{4}$$

Using Envelope gives

$$\begin{aligned} V_N(K_t, N_t) &= (1-\varepsilon-\sigma) N_t^{1-\varepsilon-\sigma} C_t^\sigma \frac{1}{N_t} + N_t^{1-\varepsilon-\sigma}\sigma C_t^{\sigma-1} w_t \\ &= N_t^{1-\varepsilon-\sigma}\sigma C_t^{\sigma-1} \left[\left(\frac{1-\varepsilon-\sigma}{\sigma}\right) \underbrace{\frac{C_t}{N_t}}_{c_t} + w_t \right] \end{aligned}$$

(plus zero terms). Forwarding one period, using the first-order condition in (4) and $N_{t+1} = n_t N_t$, we get

$$\sigma C_t^{\sigma-1}\beta_t = \alpha \underbrace{n_t^{1-\varepsilon-\sigma}}_{(N_{t+1}/N_t)^{1-\varepsilon-\sigma}} \sigma C_{t+1}^{\sigma-1} \left[\left(\frac{1-\varepsilon-\sigma}{\sigma}\right) c_{t+1} + w_{t+1} \right]$$

Next using $(C_t/C_{t+1})^{\sigma-1} = \alpha n_t^{1-\varepsilon-\sigma}(1+r_{t+1})$, we get

$$c_{t+1} = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right) [\beta_t(1+r_{t+1}) - w_{t+1}].$$

(11)

(a) For example: $n\left(\frac{Y_t}{P_t}\right) = \left(\frac{Y_t/P_t}{\bar{y}}\right)^\gamma$, where $\gamma > 0$.

(b) If choosing the same form for $n(\cdot)$ as I chose in (a), the dynamical system should look as follows:

$$\begin{aligned} A_{t+1} &= BA_t^{1-\beta}P_t^{\beta}, \\ P_{t+1} &= \left[\frac{A_t L^{1-\alpha} P_t^{\alpha-1}}{\overline{y}}\right]^{\gamma} P_t \end{aligned}$$

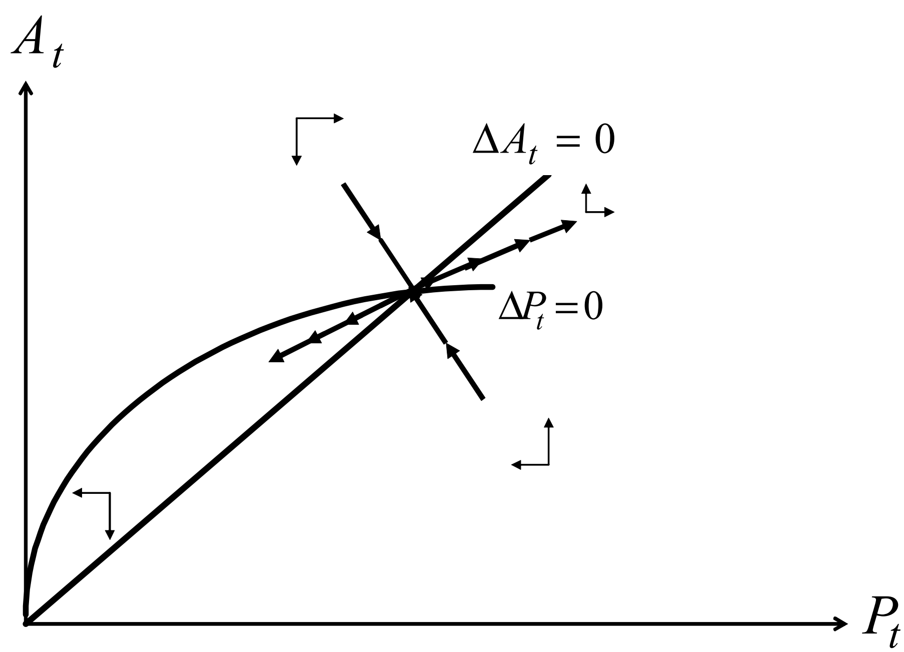
(c) The $(\Delta A_t = 0)$-locus can be written:

$$A_t = \left(B^{1/\beta}\right) P_t$$

and the $(\Delta P_t = 0)$-locus as

$$A_t = \left(\frac{\overline{y}}{L^{1-\alpha}}\right) P_t^{1-\alpha}$$

The phase diagram should look as below.



(13)

(a) $(1-\beta)kn_t = \beta\gamma c_t = \beta\gamma\left[f(x_t) - kn_t\right] \Rightarrow n_t = \frac{\beta\gamma}{1-\beta(1-\gamma)}\frac{f(x_t)}{k}$

(b) $x_{t+1} = x_t/n_t = \frac{x_t}{\frac{\beta\gamma}{1-\beta(1-\gamma)}\frac{Ax_t^\alpha}{k}} = \frac{k[1-\beta(1-\gamma)]}{\beta\gamma A}x_t^{1-\alpha}$. The steady state is stable.

(c) Setting $x_{t+1} = x_t = x^*$ we get $x^* = \left[\frac{k[1-\beta(1-\gamma)]}{\beta\gamma A}\right]^{\frac{1}{\alpha}}$. A rise in γ lowers x^* (more weight on children makes population density higher); a rise in A lowers x^* (more productive land makes population density higher); a rise in k leads to higher x^* (higher cost of rearing children leads to lower population density).

(d) FOC for n_t:

$$(1-\beta)\left[Ax_t^\alpha - kn_t\right]^{-1} k = \beta\left[\gamma(n_t)^{-1} + \Phi\frac{1}{x_{t+1}}\frac{-x_t}{n_t^2}\right].$$

Multiplying through by n_t and using $x_{t+1} = x_t/n_t$ we get $(1-\beta)\left[Ax_t^\alpha - kn_t\right]^{-1} kn_t = \beta(\gamma - \Phi)$. Solving for n_t we get

$$n_t = \left(\frac{A}{k}\right)\underbrace{\left[\frac{\beta(\gamma-\Phi)}{1-\beta-\beta\gamma+\beta\Phi}\right]}_{=\Psi}x_t^\alpha$$

(e) The value function is given by $W(x_t) = (1-\beta)\ln c_t + \beta\gamma\ln n_t + \beta W(x_{t+1})$, where c_t, n_t, and x_{t+1} are all substituted for by using the expression for optimal n_t above:

$$\begin{aligned} W(x_t) &= (1-\beta)\ln\overbrace{\{Ax_t^\alpha[1-\Psi]\}}^{=Ax_t^\alpha - kn_t = c_t} \\ &\quad +\beta\gamma\ln\overbrace{\left[\frac{\Psi A x_t^\alpha}{k}\right]}^{=n_t} \\ &\quad +\beta\underbrace{\left[\Gamma + \Phi\ln\overbrace{\left\{\frac{kx_t^{1-\alpha}}{A\Psi}\right\}}^{=x_{t+1}=x_t/n_t}\right]}_{=W(x_{t+1})}. \end{aligned}$$

This can written as a linear function of $\ln x_t$:

$$\begin{aligned} W(x_t) &= \text{constant} \\ &\quad + \underbrace{\{(1-\beta)\alpha + \beta\gamma\alpha + \beta\Phi(1-\alpha)\}}_{=\Phi} \ln x_t. \end{aligned}$$

Solving for Φ we get $\Phi = \alpha\frac{1-\beta+\beta\gamma}{1-\beta(1-\alpha)}$ which fits with Lucas' answer (p. 133).

(15)
(a) First note that

$$h_e(e_{t+1}, g_{t+1}) = \frac{g_{t+1}}{[e_{t+1} + \rho\tau + g_{t+1}]^2}.$$

Setting $G(e_{t+1}, g_{t+1}) = 0$, or $(\tau + e_{t+1})\, h_e(e_{t+1}, g_{t+1}) = h(e_{t+1}, g_{t+1})$, we get

$$\frac{(\tau + e_{t+1})\, g_{t+1}}{[e_{t+1} + \rho\tau + g_{t+1}]^2} = \frac{e_{t+1} + \rho\tau}{e_{t+1} + \rho\tau + g_{t+1}}$$

or

$$\begin{aligned} (\tau + e_{t+1})\, g_{t+1} &= [e_{t+1} + \rho\tau + g_{t+1}]\,(e_{t+1} + \rho\tau) \\ g_{t+1}\,[\tau + e_{t+1} - e_{t+1} - \rho\tau] &= [e_{t+1} + \rho\tau]\,(e_{t+1} + \rho\tau) \\ g_{t+1}\,[\tau(1-\rho)] &= [e_{t+1} + \rho\tau]^2 \end{aligned}$$

which gives $e_{t+1} = e(g_{t+1}) = \sqrt{g_{t+1}\tau(1-\rho)} - \rho\tau$. If this is something negative the non-negativity constraint on e_{t+1} binds, in which case $e_{t+1} = 0$ becomes the optimal choice. This gives the expression sought for.

(b) Set the unconstrained choice of e_{t+1} to zero; i.e., $e_{t+1} = \sqrt{g_{t+1}\tau(1-\rho)} - \rho\tau = 0$. The level of g_{t+1} at which this holds is:

$$\widehat{g} = \frac{(\rho\tau)^2}{\tau(1-\rho)} = \frac{\rho^2\tau}{1-\rho}.$$

(c)

$$h(g_{t+1}) = \begin{cases} \frac{\rho\tau}{\rho\tau + g_{t+1}} & \text{if } g_{t+1} \le \widehat{g} \\ \\ \frac{\sqrt{\tau(1-\rho)}}{\sqrt{\tau(1-\rho)} + \sqrt{g_{t+1}}} & \text{if } g_{t+1} \ge \widehat{g} \end{cases}$$

(d) $\overline{g} = \tau(1-\rho)[a(L)]^2$; $\overline{e} = \sqrt{\overline{g}\tau(1-\rho)} - \rho\tau = \tau(1-\rho)a(L) - \rho\tau$

(e) $h(\overline{e}, \overline{g}) = 1/[1 + a(L)]$

(f) $h(0, g(0, L)) = 1/[1 + a(L)]$, which is the same as in (e)

(g) When $g(0, L) = \rho\tau a(L)$ comes to exceed $\widehat{g} = \frac{\rho^2\tau}{1-\rho}$ [see (b)], i.e., when $a(L) > \rho/[1-\rho]$. Notably, this condition also ensures that $\overline{e} > 0$ and $\overline{g} > \widehat{g}$ in (d).

(16)

(a) $\lim_{P_t \to \infty} A(P) = A^*$

(b) Substitute optimal fertility, $B_t = \left(\frac{\alpha}{1+\alpha}\right)\frac{1}{v+h_t}$, into the max problem:

$$\max_{h_t \geq 0} \ln \overbrace{D\left\{1 - \frac{\alpha}{1+\alpha}\right\}(L + H_t)}^{C_t} + \alpha \ln(T_t) - \alpha \ln\left\{\left(\frac{\alpha}{1+\alpha}\right)(v + h_t)\right\} + \alpha\delta \ln[L + \underbrace{A(P_t)\,[L + H_t]\,(\rho v + h_t)}_{H_{t+1}}]$$

The first-order condition for h_t gives

$$-\alpha\left(\frac{1}{v + h_t}\right) + \alpha\delta \frac{A(P_t)\,[L + H_t]}{L + A(P_t)\,[L + H_t]\,(\rho v + h_t)} = 0$$

Solving for h_t gives

$$h_t = \frac{1}{1-\delta}\left[v(\delta - \rho) - \frac{L}{A(P_t)(L + H_t)}\right]$$

(c) As $A(P_t)(L+H_t) \to \infty$, the 2nd term in square brackets vanishes and $h_t \to \frac{v(\delta-\rho)}{1-\delta}$.

(d) Using $B_t = \left(\frac{\alpha}{1+\alpha}\right)\frac{1}{v+h_t}$ and setting $h_t = \frac{v(\delta-\rho)}{1-\delta}$ we see that B_t on the sustained growth path equals $\left(\frac{\alpha}{1+\alpha}\right)\frac{1-\delta}{v(1-\rho)}$.

(e) Use the production function for human capital:

$$H_{t+1}/H_t = A(P_t)\left[(L + H_t)/H_t\right](\rho v + h_t)\,.$$

With sustained growth in P_t and H_t it holds that $(L + H_t)/H_t \to 1$, $A(P_t) \to A^*$, and $h_t \to \frac{v(\delta-\rho)}{1-\delta}$, so that on the sustained growth path

$$\frac{H_{t+1}}{H_t} = A^* \left(\rho v + \frac{v(\delta - \rho)}{1 - \delta} \right) = \frac{A^* v \delta (1 - \rho)}{1 - \delta}$$

(f) For $T_t \to 1$ to hold human capital must grow faster than population so that H_t/P_t approaches infinity. This requires that

$$\frac{H_{t+1}}{H_t} = \frac{A^* v \delta (1 - \rho)}{1 - \delta} > \frac{P_{t+1}}{P_t} = \left(\frac{\alpha}{1 + \alpha} \right) \frac{1 - \delta}{v(1 - \rho)},$$

or

$$A^* v^2 \delta (1 - \rho)^2 > \left(\frac{\alpha}{1 + \alpha} \right) (1 - \delta)^2$$