Econ 7110 slides

Growth models: Solow, Diamond, Malthus

January 8, 2017

Production functions

Standard setting: Y=output, K (capital) and L (labor)

K, L referred to as inputs

$$Y = F(K, L) \tag{1}$$

Usually assumed to satisfy:

(1) Positive marginal products:

$$F_K(\cdot) > 0, F_L(\cdot) > 0 \tag{2}$$

(2) Diminishing marginal products:

$$F_{KK}(\cdot) < 0, F_{LL}(\cdot) < 0 \tag{3}$$

(3) The Inada condition:

$$\lim_{Z \to 0} F_Z(\cdot) = \infty \quad (4)$$
$$\lim_{Z \to \infty} F_Z(\cdot) = 0$$

for $Z = K, L$

(4) Constant Returns to Scale (CRS):

$$\lambda F(K, L) = F(\lambda K, \lambda L) \quad (5)$$

for all $\lambda > 0$

Intensive-form production function

Let lower-case variables denote per-worker levels

CRS implies

$$y = \frac{Y}{L} = \frac{F(K,L)}{L} = F\left(\frac{K}{L},1\right) = F(k,1) \equiv f(k) \tag{6}$$

Assumptions (1)-(3) imply:

$$\begin{aligned} f'(k) &> 0 \\ f''(k) &< 0 \\ \lim_{k\to 0} f'(k) &= \infty \\ \lim_{k\to\infty} f'(k) &= 0 \end{aligned} \tag{7}$$

Also, using l'Hôpital's Rule:

$$\lim_{k \to 0} \frac{f(k)}{k} = \lim_{k \to 0} \frac{f'(k)}{1} = \infty \tag{8}$$

Factor prices

Atomistic firms take factor prices as given when maximizing profits:

$$\max_{K,L} F(K,L) - \delta K - wL - rK \tag{9}$$

w=wage rate, r=real interest rate, δ=depreciation rate

$F(K,L) - \delta K$=net output (output net of capital depreciation)

Rewrite $F(K,L) = Lf(k) = Lf(\frac{K}{L})$

Exercise: show that

$$\begin{aligned} w &= F_L(K,L) = f(k) - f'(k)k \\ r &= F_K(K,L) - \delta = f'(k) - \delta \end{aligned} \tag{10}$$

Parametric examples of production functions

Cobb-Douglas:

$$\begin{aligned} F(K,L) &= K^{\alpha}L^{1-\alpha} \\ f(k) &= k^{\alpha} \end{aligned} \tag{11}$$

CES (various formulations):

$$\begin{aligned} F(K,L) &= [\alpha K^{\sigma} + (1-\alpha)L^{\sigma}]^{\frac{1}{\sigma}} \\ f(k) &= [\alpha k^{\sigma} + (1-\alpha)]^{\frac{1}{\sigma}} \end{aligned} \tag{12}$$

where $\sigma \in (-\infty, 1]$, $\sigma \neq 0$

Note: CES does not always satisfy Inada

The Solow Growth Model

Discrete time setting: the time variable t is a (non-negative) integer: $t \in \{0, 1, 2, ...\}$

Notation:

$K_t =$ capital in period t

$\delta =$ depreciation rate

$s =$ rate of saving/investment out of income, Y_t

Evolution of capital stock:

$$K_{t+1} = sY_t + (1 - \delta)K_t \tag{13}$$

L_t = population/labor force in period t

n = (net) growth rate of population

$$L_{t+1} = (1+n)L_t \tag{14}$$

Assume $n > 0$, $\delta \in (0,1]$; $s \in (0,1]$

$$Y_t = F(K_t, L_t) \tag{15}$$

Task: find difference equation for $k_t = K_t/L_t$, on the form: $k_{t+1} = \phi(k_t)$

Use (13) to (15), and $y_t = Y_t/L_t = F(k_t, 1) = f(k_t)$, to get

$$\begin{aligned} \frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) \\ = \frac{sY_t + (1-\delta)K_t}{L_t} = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t \end{aligned} \tag{16}$$

Or:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{17}$$

Properties of $\phi(k_t)$

$$
\begin{aligned}
\phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} f'(k_t) > 0 \\
\phi''(k_t) &= \tfrac{s}{1+n} f''(k_t) < 0 \\
\lim_{k_t \to 0} \phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to 0} f'(k_t) = \infty \\
\lim_{k_t \to \infty} \phi'(k_t) &= \tfrac{1-\delta}{1+n} + \tfrac{s}{1+n} \lim_{k_t \to \infty} f'(k_t) = \tfrac{1-\delta}{1+n} < 1
\end{aligned}
\tag{18}
$$

Together these guarantee: existence, uniqueness, and stability of steady state

(Uniqueness except for the trivial one where $k_t = 0$)

Illustrate in 45°-diagram

Parametric example: Cobb-Douglas production

$$
\begin{aligned}
Y_t &= Z K_t^{\alpha} L_t^{1-\alpha} \\
y_t &= Z k_t^{\alpha}
\end{aligned}
\tag{19}
$$

$$
k_{t+1} = \frac{sZk_t^{\alpha} + (1-\delta)k_t}{1+n} \equiv \phi(k_t) \tag{20}
$$

Steady state capital stock per worker:

$$
\overline{k} = \left(\frac{sZ}{n+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{21}
$$

Steady state output per worker:

$$
\overline{y} = Z^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{22}
$$

Taking stock

In this parametric example, what is effect on output per worker from rise in productivity, Z?

A rise in n?

Steady state and transition?

Implications for testing on cross-country data?

The Diamond Overlapping Generations Model

Agents live in two periods: working age, retirement

L_t = number young (working) agents in period t; $L_{t+1} = (1+n)L_t$

$c_{1,t}$ = consumption of working agent in period t

$c_{2,t}$ = consumption of retired agent in period t

s_t = saving of working agent in period t

$R_{t+1} = 1 + r_{t+1}$ = gross interest rate on savings held from period t to $t+1$

w_t = period-t wage rate

Consider agent young/working in period t

Budget constraints

$$c_{1,t} = w_t - s_t \tag{23}$$

$$c_{2,t+1} = R_{t+1} s_t \tag{24}$$

Utility:

$$U_t = U(c_{1,t}, c_{2,t+1}) \tag{25}$$

Optimal savings decision given by $s(w_t, R_{t+1})$, defined from

$$s(w_t, R_{t+1}) = \underset{s_t \in [0, w_t]}{\arg\max} U(w_t - s_t, R_{t+1} s_t) \tag{26}$$

Capital accumulation:

Total savings in period t = total capital stock in period $t+1$

$$s(w_t, R_{t+1})L_t = K_{t+1} \tag{27}$$

Recall: $L_{t+1} = (1+n)L_t$

$$\frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) = s(w_t, R_{t+1}) \tag{28}$$

Factor prices (recall) given by marginal products

$$w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t) \tag{29}$$

$$R_{t+1} = f'(k_{t+1}) + 1 - \delta \equiv R(k_{t+1}) \tag{30}$$

Thus:

$$k_{t+1} = \phi(k_t) \tag{31}$$

where $\phi(k_t)$ is defined from

$$\phi(k_t) = \frac{s\{w(k_t), R(\phi(k_t))\}}{1+n} \tag{32}$$

Cannot solve explicitly for $\phi(k_t)$

Parametric example: logarithmic utility, Cobb-Douglas production

Log utility:

$$U_t = (1-\beta)\ln(c_{1,t}) + \beta\ln(c_{2,t+1}) \tag{33}$$

First-order condition:

$$-(1-\beta)(w_t - s_t)^{-1} + \beta s_t^{-1} = 0 \tag{34}$$

Solving for s_t:

$$s_t = \beta w_t \tag{35}$$

Cobb-Douglas production:

$$\begin{aligned} Y_t &= ZK_t^{\alpha}L_t^{1-\alpha} \\ y_t &= Zk_t^{\alpha} \\ w_t &= (1-\alpha)Zk_t^{\alpha} \end{aligned} \tag{36}$$

$$k_{t+1} = \frac{\beta(1-\alpha)Zk_t^{\alpha}}{1+n} = \phi(k_t) \tag{37}$$

Steady state capital stock per worker:

$$\overline{k} = \left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{1}{1-\alpha}} \tag{38}$$

Steady state output per worker:

$$\overline{y} = Z\left[\frac{\beta(1-\alpha)Z}{1+n}\right]^{\frac{\alpha}{1-\alpha}} = Z^{\frac{1}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+n}\right]^{\frac{\alpha}{1-\alpha}} \tag{39}$$

Illustrate dynamics, steady state in 45°-diagram

Taking stock

In this parametric example, what is the effect on output per worker from rise in productivity, Z?

A rise in n?

Steady state and transition?

Implications for testing on cross-country data?

What assumptions turns Solow into Diamond in this parametric case?

The Malthus Model

Agents live in two periods: children, adults

L_t = number adult (working) agents in period t

c_t = consumption of adult agent in period t

q = cost per child (can be interpreted as consumption per child); exogenous

n_t = number of children per adult

$$L_{t+1} = n_t L_t \tag{40}$$

Note: notation is different from above!! $n_t - 1$ is here net growth rate of L_t

y_t = period-t income per adult

Y_t= total output

Production function: land, labor as inputs

$$Y_t = F(X, L_t) \tag{41}$$

X=land (here constant)

Adult's budget constraint:

$$c_t = y_t - qn_t \tag{42}$$

Utility

$$U_t = U(c_t, n_t) \tag{43}$$

Parametric example: logarithmic utility, Cobb-Douglas production

$$U_t = (1 - \beta)\ln(c_t) + \beta\ln(n_t) \tag{44}$$

Utility maximization gives

$$n_t = \left(\frac{\beta}{q}\right) y_t \tag{45}$$

$$\begin{aligned} Y_t &= Z(X)^{\alpha}(L_t)^{1-\alpha} \\ y_t &= \frac{Y_t}{L_t} = Z\left(\frac{X}{L_t}\right)^{\alpha} \end{aligned} \tag{46}$$

Dynamics equation for (adult) population:

$$L_{t+1} = n_t L_t = \left(\frac{\beta}{q}\right) Z \left(\frac{X}{L_t}\right)^{\alpha} L_t = \left(\frac{\beta}{q}\right) Z X^{\alpha} L_t^{1-\alpha} \tag{47}$$

Illustrate dynamics, steady state in 45°-diagram

Steady state population:

$$\overline{L} = \left[\left(\frac{\beta}{q} \right) Z X^{\alpha} \right]^{\frac{1}{\alpha}} = \left[\left(\frac{\beta}{q} \right) Z \right]^{\frac{1}{\alpha}} X \tag{48}$$

Steady state output per worker:

$$\overline{y} = \frac{q}{\beta} \tag{49}$$

Taking stock

In this parametric example, what is the effect on output per worker from rise in productivity, Z?

Effect on total population?

Steady state and transition?

In Solow and Diamond, we asked about effects of changes in n. Why are questions about the effects of n_t not meaningful here?

How would we test predictions of the Malthus model on data?