A simple two-sector growth model

Similar to Hansen-Prescott model

- "Malthus" sector using land, labor, capital (name a misnomer here, but use it for now)

- Solow sector using labor, capital

Here: population grows at constant rate

Production:

$$Y_{M,t} = A_{M,t} K_{M,t}^{\phi} N_{M,t}^{\mu} L^{1-\mu-\phi}$$

$$Y_{S,t} = A_{S,t} K_{S,t}^{1-\mu} N_{S,t}^{\mu}$$

$A_{i,t} =$ total factor productivity in sector i

$K_{i,t} =$ capital

$N_{i,t} =$ labor

$L =$ land (exogenous)

$i = M, S$ (Malthus, Solow)

Note: same exponent (μ) on labor in both sectors

Productivity growth same in both sectors, set to γ:

$$\begin{aligned} A_{M,t+1} &= \gamma A_{M,t}, \\ A_{S,t+1} &= \gamma A_{S,t}, \end{aligned} \tag{1}$$

Thus:

$$\frac{A_{M,t}}{A_{S,t}} = \frac{A_{M,0}}{A_{S,0}} \equiv R$$

for all t

Population dynamics given by

$$N_{t+1} = n^* N_t, \tag{2}$$

for constant $n^* \geq 1$

Later we set the value of n^* such that the wage rate in constant on pre-transition balanced growth path

$z_{N,t} =$ fraction labor in Solow sector

$z_{K,t} =$ fraction capital in Solow sector

Clearing factor markets:

$$z_{K,t} = 1 - V_t,$$

$$z_{N,t} = \frac{\phi(1 - V_t)}{\phi + (1 - \mu - \phi)\, V_t},$$

$$V_t = \min\left\{1, \left[\left(\frac{\phi}{1-\mu}\right)^{1-\mu} R\right]^{\frac{1}{1-\mu-\phi}} \frac{L}{K_t}\right\}$$

Malthus to Solow transition when low enough TFP ratio, and/or land-capital ratio

Suppose Solow sector is operated, $V_t < 1$ (holds for K_t sufficiently large)

Then total capital in Malthus sector equals

$$K_{M,t} = (1 - z_{K,t})K_t = V_t K_t = \left[\left(\frac{\phi}{1-\mu} \right)^{1-\mu} R \right]^{\frac{1}{1-\mu-\phi}} L \equiv \widehat{K}$$

which is independent of t

Follows from two assumptions: (1) same exponent (μ) on labor in both sectors; and (2) $A_{M,t}/A_{S,t} = R$ constant

Agents

Two-period OLG; no (or equal) property rights

Budget constraints

$$c_{1,t} = w_t + r_{L,t} l_t - s_t$$

$$c_{2,t+1} = (1 + r_{K,t+1}) s_t$$

Utility

$$U_t = (1 - \beta) \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

Optimal saving:

$$s_t = \beta \left[w_t + r_{L,t} l_t \right]$$

Dynamics

4 state variables that determine everything else: K_t, N_t, $A_{S,t}$, $A_{M,t}$

Last three evolve exogenously

Find dynamics of capital

Capital made up of previous period's saving

$$\begin{aligned} K_{t+1} &= s_t N_t \\ &= \beta \left[w_t + r_{L,t} l_t \right] N_t \\ &= \beta \left[w_t N_t + r_{L,t} L \right] \end{aligned}$$

where w_t and $r_{L,t}$ are given by the marginal products to labor and land:

$$w_t = \frac{\mu Y_{M,t}}{\left(1 - z_{N,t}\right) N_t}$$

$$r_{L,t} = \frac{(1 - \mu - \phi) Y_{M,t}}{L}$$

$$K_{t+1} = \beta \left(1 - \mu - \phi + \frac{\mu}{1 - z_{N,t}}\right) Y_{M,t} \tag{3}$$

Transition

The Solow sector is active when $V_t < 1$

Transition happens when K_t reaches $\widehat{K}$, defined above

Same as the post-transition level of $K_{M,t}$

Given parameter values and start values for the state variables (initial conditions), the time paths easy (in principle) to simulate

- Start with K_0, N_0, $A_{S,0}$, $A_{M,0}$

- Compute $z_{N,t}$, $z_{K,t}$, $Y_{M,t}$ for $t = 0$

- Use dynamic equations (1), (2), and (3) to update K_t, N_t, $A_{S,t}$, $A_{M,t}$ to $t = 1$

- Repeat

Calibration

The following parameter values are "free" (just guesses, not meant to target anything that the model generates)

- Parameters from producton functions: μ, ϕ
- Utility weight on second-period consumption, β
- The growth rate of productvity, γ
- Land, L

Here we set $\mu = .6$, $\phi = .1$, $\gamma = 1.05$, $\beta = .5$, and $L = 1$

We set some parameters and endogenous variables with specific targets in mind:

- We want n^* to be such that the wage rate is constant on the pre-transition balanced growth path. This implies $n^* = \gamma^{\frac{1}{1-\mu-\phi}}$ (recall Hansen-Prescott slides)

- We want $R = A_{M,t}/A_{S,t} = A_{M,0}/A_{S,0}$ to be 1

- We want the number of periods before Solow sector becomes active, denoted T, to equal 20

- We want the wage rate on pre-transition balanced growth path, denoted w^*, to be 1

Exercise: find K_0, N_0, $A_{M,0}$, $A_{S,0}$ in terms of μ, ϕ, β, R, w^*, n^*, L, and T

- Set K_0, such that the Solow sector becomes active after T periods. Use that prior to transition, K_t grows at same rate as population (n^*), and that the transition occurs when K_t reaches $\widehat{K}$

- Given K_0, set N_0 such that prior to transition: (1) K_t grows at same rate as population; (2) the marginal product of labor equals w^*

- Given K_0 and N_0, set $A_{M,0}$ such that K_t grows at same rate as population in the initial period

- Set $A_{S,0} = A_{M,0}/R$